32. Topology and beyond

Definition 32.1. *We say that a region R is simply connected if every closed curve C bounds a surface S.*

(1) $\mathbb{R}^3$ is simply connected.
(2) $\mathbb{R}^3$ minus a line is not simply connected.
(3) $\mathbb{R}^3$ minus a point is simply connected.
(4) $\mathbb{R}^3$ minus a circle is not simply connected.
(5) $\mathbb{R}^3$ minus a line segment is simply connected.

This is related to topology, which deals with the classification of geometric objects up to deforming them like pieces of rubber (so you can stretch but not tear). The surface of a sphere is topologically different from the surface of a torus. The sphere is simply connected but the torus is not.

Theorem 32.2. *If R is simply connected region in $\mathbb{R}^3$ then then $\vec{F}$ is conservative if and only if* $\operatorname{curl}\vec{F} = \vec{0}$

Proof. Suppose that $\vec{F}$ is conservative. Then $\vec{F} = \nabla f$. We have already seen that $\operatorname{curl}\vec{F}$ is then zero but it does not hurt to write down the proof again.

$$\vec{F} = \nabla f = \langle f_x, f_y, f_z\rangle.$$

Therefore

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} = (f_{zy} - f_{yz})\hat{\imath} - (f_{zx} - f_{xz})\hat{\jmath} + (f_{yx} - f_{xy})\hat{k} = \vec{0}.$$

Now suppose $\operatorname{curl}\vec{F} = \vec{0}$. Let C be a closed loop. Pick an orientable surface S which bounds C and orient S compatibly with C.

Then Stokes' theorem says

$$\oint_C \vec{F}\cdot \mathrm{d}\vec{r} = \iint_S (\operatorname{curl}\vec{F})\cdot\hat{n}\,\mathrm{d}S = 0. \qquad \square$$

Note that we have to be careful to say that S is orientable. The Möbius strip bounds a closed curve, but it is not orientable.

Suppose that S_1 and S_2 are two surfaces which bound the same curve C. Then Stokes' Theorem says

$$\iint_{S_1} \operatorname{curl}\vec{F}\cdot\hat{n}\,\mathrm{d}S = \oint_C \vec{F}\cdot \mathrm{d}r = \iint_{S_2} \operatorname{curl}\vec{F}\cdot\hat{n}\,\mathrm{d}S.$$

So the integral of the curl of $\vec{F}$ is the same for both surfaces S_1 and S_2.

In fact we can prove this in another way. Imagine constructing a new surface S by joining S_1 and S_2 along their common boundary C. Then S is a closed surface.

Note that one can reverse this process. If you start with a closed surface S and pick a closed curve C in S then you can always cut S along C to obtain two surfaces S_1 and S_2 with a common boundary C. Perhaps the easiest example of all of this is to take a sphere S and the equator C, in which case S_1 and S_2 are the upper and lower hemispheres.

Note that S_1 and S_2 must have opposite orientations to join them together; algebraically $S = S_1 - S_2$. Possibly switching S_1 and S_2 so that the orientation is outwards we can apply the divergence theorem to the closed surface S and the region it bounds D.

$$\iint_S \operatorname{curl} \vec{F} \cdot \hat{n} \, \mathrm{d}S = \iiint_D \operatorname{div}(\operatorname{curl} \vec{F}) \, \mathrm{d}V.$$

Now

$$\operatorname{div}(\operatorname{curl} \vec{F}) = \nabla \cdot (\nabla \times \vec{F}).$$

This suggests that the divergence of a curl is always zero. We check this by direct computation:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (R_y - Q_z)\hat{\imath} + (P_z - R_x)\hat{\jmath} + (Q_x - P_y)\hat{k}.$$

The divergence of this is

$$R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz} = 0.$$

So

$$\iint_{S_1 - S_2} \operatorname{curl} \vec{F} \cdot \hat{n} \, \mathrm{d}S = 0.$$

But then

$$\iint_{S_1} \operatorname{curl} \vec{F} \cdot \hat{n} \, \mathrm{d}S = \iint_{S_2} \operatorname{curl} \vec{F} \cdot \hat{n} \, \mathrm{d}S.$$

Note that the identity

$$\operatorname{div}(\operatorname{curl} \vec{F}) = 0,$$

is very useful on its own.

Note that one can compose in the opposite direction. If f is a scalar function then ∇f is a vector field, and we can take the curl of this.

$$\operatorname{curl}(\nabla f) = \nabla \times (\nabla f) = \vec{0}.$$

We have already seen that this is the zero vector. This has an interesting physical interpretation. Recall that $\operatorname{curl} \vec{F}$ measures the rotation component of a vector field. So the fact that $\operatorname{curl}(\nabla f) = 0$ says that every force field given by a potential imparts no rotation component. For example gravity imparts no rotation.