3. Matrices

Often if one starts with a coordinate system (x_1, x_2, x_3), sometimes it is better to work in a coordinate system (y_1, y_2, y_3) related to the old coordinate system in a simple way:

$$\begin{aligned} 2x_1 - x_2 + x_3 &= y_1 \\ -3x_1 + x_2 + 4x_3 &= y_2 \\ 2x_1 - x_2 + x_3 &= y_3. \end{aligned}$$

Matrices are simply a way to encode this transformation in a compact form

$$\begin{pmatrix} 2 & -1 & 1 \\ -3 & 1 & 4 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

In even more compact notation, $A\vec{x} = \vec{y}$, where A is a 3×3 matrix, $\vec{x}$ is a column vector, a 3×1 matrix (3 rows, 1 column) and $\vec{y}$ has the same shape. To get the entries of the product $A\vec{x}$ take the dot product of a row from A and a column from $\vec{x}$.

More generally, if we want to multiply two matrices A and B, take the dot product of the rows of A and the columns of B:

$$\begin{pmatrix} 2 & 3 & 1 & 2 \\ -1 & -1 & 3 & 4 \\ 0 & -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 3 \\ -3 & -2 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} * & * \\ * & * \\ * & x \end{pmatrix}$$

Question 3.1. *What is x?*

It is the entry obtained by taking the dot product of the 3rd row of A and the 2nd column of B:

$$x = \langle 0, -1, 1, 1 \rangle \cdot \langle 1, 3, -2, 0 \rangle = 0 - 3 - 2 + 0 = -5.$$

For the product AB to make sense, A must have the same number of columns as B has rows.

Question 3.2. *Is $AB = BA$ in general?*

No, for four different reasons.

Sometimes the product make sense one way but not the other way. For example if A is 4×2 and B is 2×3 the product AB is a 4×3 matrix but the product BA does not make sense (3 does not match 4).

Sometimes the product makes sense both ways but the shape is different. For example if A is 3×1 and B is 1×3, AB is a 3×3 (consisting of the nine dot products obtained by multiplying an entry of A with an entry of B). But BA has shape 1×1, one dot product in $\mathbb{R}^3$.

If A and B are both square the product makes sense both ways and has the same shape, but it is still not the same. For example, take

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Then the first entry of AB is 4 but the first entry of BA is 1.

Finally, it is clear that matrix multiplication does not commute if one thinks about transformations. E.g. if A corresponds to reflection in the y-axis and B to rotation through $\pi/4$, then AB represents rotation through $\pi/4$ and reflection in the y-axis and BA represents reflection in the y-axis followed by rotation through $\pi/4$.

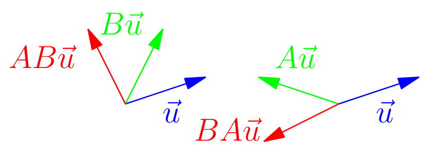

FIGURE 1. AB vs BA

There is one very special square matrix, called the identity matrix. For example,

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

As a transformation, I_3 does not do anything. In fact,

$$I_3 B = B \qquad \text{and} \qquad AI_3 = A,$$

whenever these products make sense.

Question 3.3. *What does the matrix*

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

do as a transformation?

Well it replaces the vector $\langle a_1, a_2 \rangle$ by the vector $\langle a_2, a_1 \rangle$. So $A\hat{\imath} = \hat{\jmath}$ and $A\hat{\jmath} = \hat{\imath}$. A represents reflection in the line $y = x$. So $A^2 = I_2$.

Probably the most important property of the determinant of a matrix is the following

Theorem 3.4. *Let A and B be square matrices of the same size. Then*

$$\det(AB) = \det A \det B.$$

Some transformations are reversible. If A represents a reversible transformation, the matrix corresponding to the inverse transformation is called the inverse matrix A^{-1}. We have

$$A\vec{x} = \vec{y} \qquad \text{and} \qquad \vec{x} = A^{-1}\vec{y},$$

and

$$A^{-1}A = I \qquad \text{and} \qquad AA^{-1} = I.$$

In words the inverse matrix undoes the effect of A.

There is a very useful characterisation of which matrices are invertible (have inverses):

Theorem 3.5. *A is invertible if and only if* $\det A \neq 0$.

Here is a recipe for calculating the inverse of a matrix. This recipe is perfect for 2×2 matrices, (barely) acceptable for 3×3 matrices and simply diabolical for anything larger.

First 2×2. If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{then} \qquad A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

In general, one adopts the following procedure, which we illustrate with the following 3×3 matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 12 \\ 5 & 6 & 0 \end{pmatrix}.$$

Step 1: Form the matrix of minors. In the (i, j) entry, put the determinant you get by erasing the ith row and jth column of A:

$$\begin{pmatrix} -72 & -60 & -15 \\ -18 & -15 & -4 \\ 15 & 12 & 3 \end{pmatrix}.$$

For example the entry in the second row, third column is obtained by taking the matrix A and deleting the second row and third column to get the matrix

$$\begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix}.$$

Now just take the determinant of this 2×2 matrix

$$\begin{vmatrix} 1 & 2 \\ 5 & 6 \end{vmatrix} = 6 - 10 = -4.$$

Step 2: Now flip the signs of the matrix of minors according to the following pattern:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

The result is the matrix of cofactors:

$$\begin{pmatrix} -72 & 60 & -15 \\ 18 & -15 & 4 \\ 15 & -12 & 3 \end{pmatrix}.$$

Step 3: Now take the transpose (flip the matrix about its main diagonal) to get the adjoint matrix:

$$\mathrm{Adj}(A) = \begin{pmatrix} -72 & 18 & 15 \\ 60 & -15 & -12 \\ -15 & 4 & 3 \end{pmatrix}.$$

Step 4: Divide by the determinant to get the inverse matrix. In our case the determinant is 3. So we divide by 3,

$$A^{-1} = \frac{1}{3}\begin{pmatrix} -72 & 18 & 15 \\ 60 & -15 & -12 \\ -15 & 4 & 3 \end{pmatrix} = \begin{pmatrix} -24 & 6 & 5 \\ 20 & -5 & -4 \\ -5 & 4/3 & 1 \end{pmatrix}.$$

Finally, let's check that this is the right answer. We should have

$$A^{-1}A = \begin{pmatrix} -24 & 6 & -5 \\ 20 & -5 & 4 \\ -5 & 4/3 & -1 \end{pmatrix} \begin{pmatrix} -72 & -60 & -15 \\ -18 & -15 & -4 \\ 15 & 12 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3.$$

Let's pick an entry at random. Let's calculate the entry in the 2nd row, 3rd column. We should get 0. In fact we get the dot product of the 2nd row of the first matrix and the 3rd column of the second matrix:

$$\langle 20, -5, 4\rangle \cdot \langle -15, -4, 3\rangle = 0,$$

which is indeed correct.