28. How to compute the flux

Let's start with the case when S is the graph of a function $z = f(x, y)$ lying over a region R of the plane. If we have a small rectangle with sides Δx and Δy in R then in space we roughly we get a parallelogram with vertices

$$\begin{array}{rl} (x, y, f(x, y)) & \quad (x + \Delta x, y, f(x + \Delta x, y)) \\ (x, y + \Delta y, f(x, y + \Delta y)) & \quad (x + \Delta x, y + \Delta y, f(x + \Delta x, y + \Delta y)). \end{array}$$

By linear approximation,

$$f(x+\Delta x, y) \approx f(x, y)+f_x(x, y)\Delta x \qquad \text{and} \qquad f(x, y+\Delta y) \approx f(x, y)+f_y(x, y)\Delta y.$$

and so on. So we have a parallelogram with two sides

$$\vec{v} = \langle \Delta x, 0, f_x(x, y)\Delta x\rangle = \Delta x\langle 1, 0, f_x\rangle \qquad \text{and} \qquad \vec{w} = \langle 0, \Delta y, f_y(x, y)\Delta y\rangle = \Delta y\langle 0, 1, f_y\rangle.$$

The cross product is both a vector normal to the base of the parallelogram and has length the area of the parallelogram. We have

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = \hat{\imath}\begin{vmatrix} 0 & f_x \\ 1 & f_y \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 1 & f_x \\ 0 & f_y \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -f_x\hat{\imath} - f_y\hat{\jmath} + \hat{k}.$$

It follows that

$$\Delta\vec{S} \approx \vec{v} \times \vec{w} = \Delta x\Delta y\langle -f_x, -f_y, 1\rangle.$$

Taking the limit as Δx and Δy go to zero, we get

$$\mathrm{d}\vec{S} = \langle -f_x, -f_y, 1\rangle \, \mathrm{d}x \, \mathrm{d}y.$$

We can use this to recover

$$\hat{n} = \frac{1}{\sqrt{1 + f_x^2 + f_y^2}}\langle -f_x, -f_y, 1\rangle \qquad \text{and} \qquad \mathrm{d}S = |\mathrm{d}\vec{S}| = \sqrt{1 + f_x^2 + f_y^2}\, \mathrm{d}x \, \mathrm{d}y.$$

In practice, it is usually better not to find the separate pieces.

Question 28.1. *Find the flux of $\vec{F} = z\hat{k}$ across the surface S given by the paraboloid $z = x^2 + y^2$ above the circle R in the xy-plane, given by $x^2 + y^2 \leq 1$, oriented so the normal points upwards (which is into the paraboloid).*

$$\mathrm{d}\vec{S} = \langle -2x, -2y, 1\rangle \, \mathrm{d}x \, \mathrm{d}y.$$

Hence

$$\vec{F} \cdot \mathrm{d}\vec{S} = z \, \mathrm{d}x \, \mathrm{d}y.$$

So the flux is

$$\iint_S \vec{F}\cdot \mathrm{d}\vec{S} = \iint_R z\,\mathrm{d}x\,\mathrm{d}y = \int_0^{2\pi}\int_0^1 r^3\,\mathrm{d}r\,\mathrm{d}\theta.$$

The inner integral is

$$\int_0^1 r^3\,\mathrm{d}r = \left[\frac{r^4}{4}\right]_0^1 = \frac{1}{4}.$$

The outer integral is

$$\int_0^{2\pi} \frac{1}{4}\,\mathrm{d}\theta = \frac{\pi}{2}.$$

More generally, suppose S is given as a parametric surface $x(u, v)$, $y(u, v)$ and $z(u, v)$, then we can integrate using u and v, so that $\vec{r} = \vec{r}(u, v)$. Arguing as above,

$$\mathrm{d}\vec{S} = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\,\mathrm{d}u\,\mathrm{d}v.$$

Apart from parametrisations, a surface S might be given by a constraint. S might be given implicitly, by an equation $g(x, y, z) = 0$. In this case

$$\vec{N} = \nabla g = \langle g_x, g_y, g_z\rangle,$$

is normal to S and so

$$\hat{n} = \frac{\vec{N}}{|\vec{N}|}$$

is a unit normal.

To get ΔS, consider the projection to the xy-plane (assume that the plane is not vertical; if it is vertical, just project onto the xz-plane or the yz-plane). The key point is to figure out how area changes under projection. I claim

$$\Delta A = \cos\alpha \Delta S,$$

where α is the angle of the surface with the horizontal, that is, the angle between $\vec{N}$ and the vertical $\hat{k}$. The reason for this is the same reason that the projection of a circle, lying in a slanted plane, is an ellipse. Note that every plane contains one horizontal line. To figure out how area changes under projection, one can rotate the plane so that this line is the y-axis. So lengths in the y direction are unchanged. In the x-direction, one gets a right angled triangle. The original length is a hypotenuse and the new length is the adjacent. So lengths in the x-direction scale by $\cos\alpha$. In total the area scales by $\cos\alpha$. But

$$\vec{N}\cdot\hat{k} = |N|\cos\alpha.$$

Putting all of this together,

$$\mathrm{d}\vec{S} = \frac{\vec{N}}{|\vec{N}.\hat{k}|}\,\mathrm{d}x\,\mathrm{d}y.$$

Example 28.2. *Suppose that*

$$g(x,y,z) = z - f(x,y),$$

so that $g = 0$ defines the graph of f. Then

$$\vec{N} = \nabla g = \langle -f_x, -f_y, 1\rangle \qquad \textit{and} \qquad \vec{N}\cdot\hat{k} = 1,$$

so we get the old formula.

Theorem 28.3 (Divergence Theorem)**.** *Let S be a closed surface bounding a solid D, oriented outwards. Let $\vec{F}$ be a vector field with continuous partial derivatives. Then*

$$\iint_S \vec{F}\cdot\mathrm{d}\vec{S} = \iiint_D \operatorname{div}\vec{F}\,\mathrm{d}V \qquad \textit{where} \qquad \operatorname{div}\vec{F} = P_x + Q_y + R_z.$$

This has the same physical interpretation as before. The total amount of material leaving S is equal to the amount of material created (or destroyed) inside the solid D.

Example 28.4. *Let $\vec{F} = z\hat{k}$ and let S be the surface of a sphere of radius a.*

$$\operatorname{div}\vec{F} = 0 + 0 + 1 = 1,$$

and so

$$\iint_S \vec{F}\cdot\mathrm{d}\vec{S} = \iiint_D \operatorname{div}\vec{F}\,\mathrm{d}V = \iiint_D 1\,\mathrm{d}V = \frac{4}{3}\pi a^3.$$

It is convenient to introduce some symbolic notation.

$$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle$$

is called the del operator.

$$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\rangle$$

is the gradient. We have

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle\cdot\langle P, Q, R\rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

is the divergence.