25. Review

Double integrals Integrate function $f(x,y)$ over a region R:

$$\iint_R f \, \mathrm{d}A.$$

Computes the volume of the graph of f lying over R.

Example 25.1. *Evaluate*

$$\int_0^1 \int_0^{x^2} \frac{xe^y}{1-y} \, \mathrm{d}y \, \mathrm{d}x.$$

We cannot caculate this directly.

First we figure out the region of integration. $0 \leq x \leq 1$. Given x, we have $0 \leq y \leq x^2$. So we have the region R between $x = 0$ and $x = 1$ under the graph of $y = x^2$. Then we switch the order of integration.

$$\int_0^1 \int_0^{x^2} \frac{xe^y}{1-y} \, \mathrm{d}y \, \mathrm{d}x = \iint_R \frac{xe^y}{1-y} \, \mathrm{d}y \, \mathrm{d}x = \int_0^1 \int_{\sqrt{y}}^1 \frac{xe^y}{1-y} \, \mathrm{d}x \, \mathrm{d}y.$$

The inner integral is

$$\int_{\sqrt{y}}^1 \frac{xe^y}{1-y} \, \mathrm{d}x = \left[\frac{x^2 e^y}{2(1-y)} \right]_{\sqrt{y}}^1 = \frac{e^y(1-y)}{2(1-y)} = \frac{1}{2}e^y.$$

So the outer integral is

$$\int_0^1 \frac{1}{2}e^y \, \mathrm{d}x = \left[\frac{1}{2}e^y \right]_0^1 = \frac{e-1}{2}.$$

We can use the double integral to calculate the mass, centre of mass and moment of inertia:

Example 25.2. *A metal plate is in the shape of a circle of radius* 20*cm. Its density in g/cm*^{2} *at a distance of* r *cm from the centre of the circle is* $10r + 3$*.*

Find the total mass as an integral.

$$M = \iint_R \delta \, \mathrm{d}A = \int_0^{2\pi} \int_0^{20} (10r+3) r \, \mathrm{d}r \, \mathrm{d}\theta.$$

Line integrals Integrate a vector field $\vec{F}$ over an oriented curve C.

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r}.$$

Represents the work done.

One can compute directly, by parametrising C. Let $C = C_1+C_2+C_3$ be the curve which starts at $(0,0)$ goes along the x-axis to $(1,0)$, goes around the unit circle until $(0,1)$ and comes back to the origin.

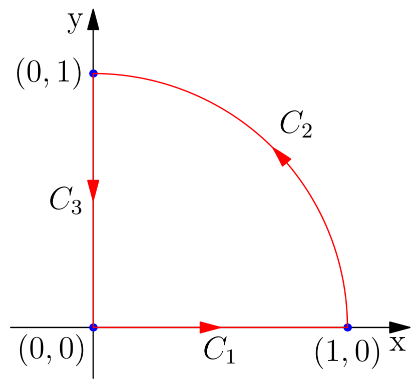

FIGURE 1. The curve C

Let $\vec{F} = -x^3\hat{\imath} + x^2y\hat{\jmath}$.

$$\oint_C \vec{F}\cdot \mathrm{d}\vec{r} = \int_{C_1} \vec{F}\cdot \mathrm{d}\vec{r} + \int_{C_2} \vec{F}\cdot \mathrm{d}\vec{r} + \int_{C_3} \vec{F}\cdot \mathrm{d}\vec{r}.$$

Note that

$$\int_{C_3} \vec{F}\cdot \mathrm{d}\vec{r} = 0,$$

as $\vec{F} = \vec{0}$ along the y-axis. Parametrise C_1 by $x(t) = t$, $y(t) = 0$.

$$\vec{F} = \langle -t^3, 0\rangle \qquad \text{and} \qquad \mathrm{d}\vec{r} = \langle 1, 0\rangle\,\mathrm{d}t.$$

So

$$\int_{C_1} \vec{F}\cdot \mathrm{d}\vec{r} = \int_0^1 \langle -t^3, 0\rangle \cdot \langle 1, 0\rangle\,\mathrm{d}t = \int_0^1 -t^3\,\mathrm{d}t = \left[-\frac{1}{4}t^4\right]_0^1 = -\frac{1}{4}.$$

Parametrise C_2 by $x(t) = \cos t$, $y(t) = \sin t$.

$$\vec{F} = \langle -\cos^3 t, \cos^2 t \sin t\rangle \qquad \text{and} \qquad \mathrm{d}\vec{r} = \langle -\sin t, \cos t\rangle\,\mathrm{d}t.$$

So

$$\begin{aligned}\int_{C_1} \vec{F}\cdot \mathrm{d}\vec{r} &= \int_0^{\pi/2} \langle -\cos^3 t, \cos^2 t \sin t\rangle \cdot \langle -\sin t, \cos t\rangle\,\mathrm{d}t \\ &= \int_0^{\pi/2} 2\cos^3 t \sin t\mathrm{d}t \\ &= \left[-\cos^4 t/2\right]_0^{\pi/2} = 1/2.\end{aligned}$$

In total we get 1/4. We can also use Green's theorem:

$$\begin{aligned}\oint_C \vec{F}\cdot \mathrm{d}\vec{r} &= \iint_R \operatorname{curl}\vec{F}\,\mathrm{d}A\\ &= \int_0^{\pi/2}\int_0^1 r^3\cos\theta\,\mathrm{d}r\mathrm{d}\theta.\end{aligned}$$

The inner integral is

$$\int_0^1 r^3\cos\theta\,\mathrm{d}r = \left[\frac{1}{4}r^4\cos\theta\right]_0^1 = \frac{1}{4}\cos\theta.$$

So the outer integral is

$$\int_0^{\pi/2}\frac{1}{4}\cos\theta\,\mathrm{d}\theta = \left[\frac{1}{4}\sin\theta\right]_0^{\pi/2} = \frac{1}{4}.$$

What about the same question, but now let us compute the flux.

$$\oint_C \vec{F}\cdot\hat{n}\,\mathrm{d}s = \int_{C_1}\vec{F}\cdot\hat{n}\,\mathrm{d}s + \int_{C_2}\vec{F}\cdot\hat{n}\,\mathrm{d}s + \int_{C_3}\vec{F}\cdot\hat{n}\,\mathrm{d}s.$$

Once again the flux across C_3 is zero. Along C_1 the normal vector is $-\hat{\jmath}$. So the flux is zero, since $\vec{F}$ is parallel to $\hat{\imath}$ along the x-axis. Along C_2, we have

$$\hat{n}\,\mathrm{d}s = \langle \mathrm{d}y, -\mathrm{d}x\rangle.$$

So

$$\begin{aligned}\int_{C_1}\vec{F}\cdot\hat{n}\mathrm{d}s &= \int_0^{\pi/2}\langle -\cos^3 t, \cos^2 t\sin t\rangle\cdot\langle\cos t,\sin t\rangle\,\mathrm{d}t\\ &= \int_0^{\pi/2} -\cos^4 t + \cos^2 t\sin^2 t\,\mathrm{d}t\\ &= \frac{-\pi}{8}.\end{aligned}$$

Or we could apply the normal form of Green's theorem:

$$\begin{aligned}\oint_C \vec{F}\cdot\hat{n}\,\mathrm{d}s &= \iint_R \operatorname{div}\vec{F}\,\mathrm{d}A\\ &= \iint_R -2x^2\,\mathrm{d}A\\ &= \int_0^{\pi/2}\int_0^1 -2r^3\cos^2\theta\,\mathrm{d}r\mathrm{d}\theta.\end{aligned}$$

The inner integral is

$$\int_0^1 -2r^3\cos\theta\,\mathrm{d}r = \left[-\frac{1}{2}r^4\cos^2\theta\right]_0^1 = -\frac{1}{2}\cos^2\theta.$$

So the outer integral is

$$\int_0^{\pi/2} -\frac{1}{2}\cos^2\theta \,\mathrm{d}\theta = \left[-\frac{t}{4} - \frac{1}{8}\sin(2\theta) \right]_0^{\pi/2} = -\frac{1}{8}\pi.$$

Let

$$\vec{F} = (3x^2 - 2y\sin x\cos x)\hat{\imath} + (a\cos^2 x + 1)\hat{\jmath}.$$

For which values of a is $\vec{F}$ a gradient vector field?

$$M_y = -2\sin x \qquad \text{and} \qquad N_x = -2a\cos x\sin x.$$

These are equal if and only if $a = 1$. For this value of a, what is the integral over the curve C,

$$x(t) = t^2 \qquad \text{and} \qquad y(t) = t^3 - 1,$$

$0 \leq t \leq 1$?

Find a potential function $f(x, y)$. We want

$$f_x = 3x^2 - 2y\sin x\cos x \qquad \text{and} \qquad f_y = \cos^2 x + 1.$$

Integrate the first equation with respect to x,

$$f(x, y) = x^3 - y\cos^2 x + g(y).$$

Use the second equation to determine $g(y)$,

$$-\cos^2 x + \frac{dg}{dy} = \cos^2 x + 1 \qquad \text{so that} \qquad \frac{dg}{dy} = 1.$$

Hence $g(y) = y + c$. So

$$f(x, y) = x^3 - y\cos^2 x + y,$$

will do.

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_C \nabla f \cdot \mathrm{d}\vec{r} = f(1, 1) - f(0, 0) = 1.$$