22. Greens Theorem

Theorem 22.1 (Green's Theorem). *If C is a positively oriented closed curve enclosing a region R then*

$$\oint_C \vec{F}\cdot \mathrm{d}\vec{r} = \iint_R \operatorname{curl}\vec{F}\,\mathrm{d}A.$$

The circle in the centre of the integral sign is simply to emphasize that the line integral is around a closed loop. Here C is oriented so that R is on the left as we go around C.

Green's Theorem, in the language of differentials, comes out as

$$\oint_C M\,\mathrm{d}x + N\,\mathrm{d}y = \iint_R (N_x - M_y)\,\mathrm{d}A.$$

For example, let C be a unit circle centred at $(2, 0)$, oriented counterclockwise and let R be the unit disk, centred at $(2, 0)$.

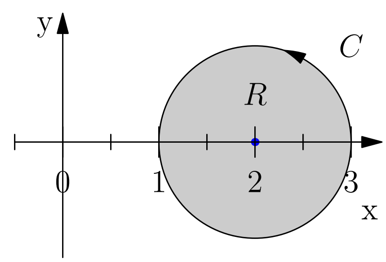

Figure 1. The region R with boundary C

We have

$$\begin{aligned}\oint_C ye^{-x}\,\mathrm{d}x + \left(\frac{1}{2}x^2 - e^{-x}\right)\,\mathrm{d}y &= \oint_C M\,\mathrm{d}x + N\,\mathrm{d}y\\ &= \iint_R (N_x - M_y)\,\mathrm{d}A\\ &= \iint_R (x + e^{-x} - e^{-x})\,\mathrm{d}A\\ &= \iint_R x\,\mathrm{d}A.\end{aligned}$$

Now one could calculate the last integral by direct calculation of the iterated integral. On the other hand, if we divide the last integral by the area, we get $\bar{x}$, the x-coordinate of the centre of mass. Obviously the centre of mass is at the centre of the circle, so $\bar{x} = 2$. The area is π, so the integral is 2π.

Corollary 22.2. *Let $\vec{F} = M\hat{\imath} + N\hat{\jmath}$ be a vector field which is defined and differentiable on the whole of $\mathbb{R}^2$.*

Then $\vec{F}$ is a gradient vector field if and only if $M_y = N_x$.

Proof. Suppose that $M_y = N_x$. Then $\operatorname{curl} \vec{F} = 0$. By (**??**), we have

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_R \operatorname{curl} \vec{F} \, \mathrm{d}A = \iint_R 0 \, \mathrm{d}A = 0.$$

Hence $\vec{F}$ is conservative. □

Note that this only works if the region R is completely contained in the locus where $\vec{F}$ is defined. In question B5 of last weeks hwk, the integral around the unit circle the vector field $\vec{F}$ is not defined at the origin.

We now describe the proof of (**??**)

Proof of (**??**). First a couple of useful reduction steps. For a start it suffices to prove two separate identities:

$$\oint_C M \, \mathrm{d}x = \iint_R -M_y \, \mathrm{d}A \qquad \text{and} \qquad \oint_C N \, \mathrm{d}y = \iint_R N_x \, \mathrm{d}A.$$

To get the general result, add these two identities.

Secondly, if R is the union of two regions R_1 and R_2 and we know the result for both regions R_1 and R_2 then we know it for R. Indeed,

$$\begin{aligned}\oint_C \vec{F} \cdot \mathrm{d}\vec{r} &= \oint_{C_1} \vec{F} \cdot \mathrm{d}\vec{r} + \oint_{C_2} \vec{F} \cdot \mathrm{d}\vec{r} \\ &= \iint_{R_1} \operatorname{curl} \vec{F} \, \mathrm{d}A + \iint_{R_2} \operatorname{curl} \vec{F} \, \mathrm{d}A \\ &= \iint_R \operatorname{curl} \vec{F} \, \mathrm{d}A.\end{aligned}$$

Here C is the boundary of R and C_1, C_2 are the boundaries of R_1 and R_2. The first equality is therefore a little bit more subtle than might first appear; the key thing is that we might get some cancelling.

Using these two reduction steps, we get down to the kernel of the proof. Prove that

$$\oint_C M \, \mathrm{d}x = \iint_R -M_y \, \mathrm{d}A$$

where R is a vertically simple region, that is a region of the form

$$a \leq x \leq b \qquad \text{and} \qquad f_0(x) \leq y \leq f_1(x).$$

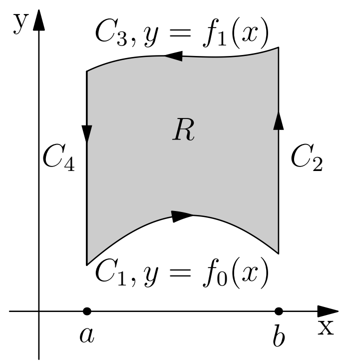

FIGURE 2. Typical vertically simple region

So R is the region between the graph of two functions. Now we calculate both sides. For the LHS break C into four pieces,

$$C = C_1 + C_2 + C_3 + C_4,$$

where C_1 is the lower edge, the graph of $y = f_0(x)$ between a and b, C_2 is the right vertical segment, C_3 is the upper edge, the graph of $y = f_1(x)$ between a and b, and C_4 is the left vertical segment. Now

$$\int_{C_2} M \, \mathrm{d}x = \int_{C_4} M \, \mathrm{d}x = 0,$$

since x is constant on these edges. For the other two edges use the parametrisation $x(t) = t$, $y(t) = f_0(t)$, $a \leq t \leq b$ and $x(t) = t$, $y(t) = f_1(t)$, $a \leq t \leq b$, but with the opposite orientation, so that we get

$$\oint_C M \, \mathrm{d}x = \int_{C_1} M \, \mathrm{d}x + \int_{C_3} M \, \mathrm{d}x = \int_a^b M(t, f_0(t)) \, \mathrm{d}t - \int_a^b M(t, f_1(t)) \, \mathrm{d}t.$$

For the RHS we have

$$-\iint_R M_y \, \mathrm{d}A = -\int_a^b \int_{f_0(x)}^{f_1(x)} M_y \, \mathrm{d}y \, \mathrm{d}x.$$

Now the inner integral is

$$\int_{f_0(x)}^{f_1(x)} M_y \, \mathrm{d}y = -\int_a^b M(x, f_1(x)) - M(x, f_0(x)) \, \mathrm{d}x,$$

and so the outer integral is

$$\int_a^b M(x, f_0(x)) - M(x, f_1(x)) \, \mathrm{d}x,$$

the same as the LHS.

There is a similar calculation with N replacing M, horizontally simple regions replacing vertically simple regions and suitable switching of x and y. □

Example 22.3. *The area of a region R can be evaluated using Green's theorem. For example,*

$$\operatorname{area}(R) = \iint_R 1 \, \mathrm{d}A = \oint_C x \, \mathrm{d}y.$$

One can actually build physical devices that measure area this way. If one has a figure on a piece of paper, a planimeter can be used to find the area. Move the end of the planimeter so it traces out the curve C. At the end one can read off the area.

For a linear planimeter, there is an arm AB. B is constrained to lie in the y-axis and the point A traces out the curve C. Suppose it has coordinates $(0, b)$. Suppose the coordinates of A are (x, y). So $\overrightarrow{AB} = \langle x, y-b\rangle$. It follows that if $\vec{F} = \langle b-y, x\rangle$, then $\vec{F}$ is perpendicular to $\overrightarrow{AB}$. The length of $\vec{F}$ is a constant equal to the length m of the arm. By (**??**)

$$\begin{aligned}
\oint_C M \, \mathrm{d}x + N \, \mathrm{d}y &= \iint_R (N_x - M_y) \, \mathrm{d}A \\
&= \iint_R 1 - \frac{\partial(b-y)}{\partial y} \, \mathrm{d}A \\
&= \iint_R 1 \mathrm{d}A \\
&= \operatorname{area}(R).
\end{aligned}$$

Here we used the fact that

$$\frac{\partial(b-y)}{\partial y} = \frac{\partial\sqrt{m^2 - x^2}}{\partial y} = 0.$$