18. Change of variables

Question 18.1. *What is the area of the ellipse*

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1?$$

The area is

$$\begin{aligned}\iint_R 1\,\mathrm{d}A &= \iint_{\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2\leq 1} 1\,\mathrm{d}x\mathrm{d}y \\ &= \iint_{u^2+v^2\leq 1} ab\,\mathrm{d}u\mathrm{d}v \\ &= \pi ab.\end{aligned}$$

Here we changed variable from x and y to $u = x/a$ and $v = y/b$. We have

$$\mathrm{d}u = \frac{\mathrm{d}x}{a} \qquad \text{and} \qquad \mathrm{d}v = \frac{\mathrm{d}y}{b}.$$

It follows that

$$\mathrm{d}u\,\mathrm{d}v = \frac{1}{ab}\mathrm{d}x\,\mathrm{d}y.$$

How about if the change of variables is more complicated?

To warm up, let's consider a linear transformation.

$$\begin{aligned}u &= 2x - y \\ v &= x + y.\end{aligned}$$

In this case, a rectangle in the xy-plane gets mapped to a parallelogram. In terms of matrices,

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

It follows that the square given by $\hat{\imath}$ and $\hat{\jmath}$ gets mapped to the parallelogram with sides $2\hat{\imath} + \hat{\jmath} = \langle 2, 1\rangle$ and $-\hat{\imath} + \hat{\jmath} = \langle -1, 1\rangle$. The area of this parallelogram is the absolute value of the determinant:

$$\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 3.$$

So

$$\mathrm{d}u\,\mathrm{d}v = 3\mathrm{d}x\,\mathrm{d}y.$$

(Since the map is linear, every rectangle gets rescaled by the same factor of 3).

In general, by the approximation formula,

$$\begin{aligned}\Delta u &\approx u_x\Delta x + u_y\Delta y \\ \Delta v &\approx v_x\Delta x + v_y\Delta y.\end{aligned}$$

In terms of matrices

$$\begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix} \approx \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}.$$

Then a small rectangle in the xy-plane gets mapped approximately to a parallelogram of area the absolute value of the determinant

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}.$$

The determinant is called the Jacobian,

$$J = \frac{\partial(u, v)}{\partial(x, y)}.$$

Taking the limit as Δx and Δy go to zero, we get

$$\mathrm{d}u \, \mathrm{d}v = |J| \mathrm{d}x \, \mathrm{d}y.$$

Note that we take the absolute value, as area is always positive.

Let's see what happens if we go from Cartesian coordinates to polar.

$$x = r\cos\theta$$
$$y = r\sin\theta.$$

The determinant is

$$\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r,$$

so that

$$\mathrm{d}x \, \mathrm{d}y = r\mathrm{d}r \, \mathrm{d}\theta,$$

as expected.

Question 18.2. *Let R be the square with vertices $(\pm 1, 0)$, $(0, \pm 1)$. What is*

$$\iint_R \frac{\sin^2(x - y)}{x + y + 2} \, \mathrm{d}x\mathrm{d}y?$$

Let's change coordinates to $u = x - y$ and $v = x + y$. Note that this has two benefits. The integrand simplifies and the sides of the square are given by u or v constant. The side from $(0, 1)$ to $(1, 0)$ corresponds to $v = 1$. u ranges from -1 to 1. Similarly the four sides are $u = \pm 1$ and $v = \pm 1$. The Jacobian is

$$J = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2.$$

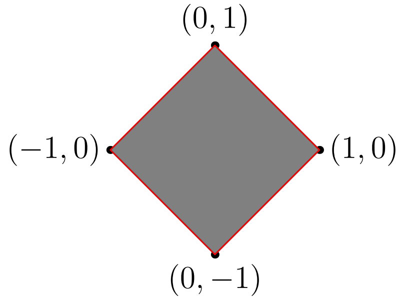

FIGURE 1. Region of integration

So

$$\mathrm{d}u\,\mathrm{d}v = 2\mathrm{d}x\,\mathrm{d}y.$$

$$\iint_R \frac{\sin^2(x-y)}{x+y+2}\,\mathrm{d}x\mathrm{d}y = \int_{-1}^{1}\int_{-1}^{1}\frac{1}{2}\frac{\sin^2 u}{v+2}\,\mathrm{d}u\mathrm{d}v.$$

It is then straightforward to finish off.

One more example. Let's compute

$$\int_0^1\int_0^1 x^2y\,\mathrm{d}x\mathrm{d}y,$$

by using the change of variable $u = x$ and $v = xy$. The Jacobian is the absolute value of

$$\begin{vmatrix} 1 & 0 \\ y & x \end{vmatrix} = x.$$

Note that x is positive over the square, so no need to take the absolute value.

$$x^2y\,\mathrm{d}x\mathrm{d}y = x^2y\frac{1}{x}\,\mathrm{d}u\mathrm{d}v = v\,\mathrm{d}u\mathrm{d}v.$$

Now we figure out the range of integration. First the outer limits. What is the maximum value of v over the square? Well 1, achieved at the point $(1,1)$. And the minimum value is 0, achieved at $(0,0)$. So v ranges from 0 to 1. What about u? Well if we fix a value of v, we get a hyperbola. The maximum value of $u = x$ is always 1. We have $xy = v$. The minimum value is when $x = v$.

So the integral in uv-coordinates is

$$\int_0^1\int_0^1 x^2y\,\mathrm{d}x\mathrm{d}y = \int_0^1\int_v^1 v\,\mathrm{d}u\mathrm{d}v.$$

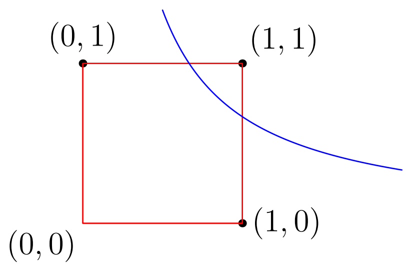

FIGURE 2. Limits for $u = x$