16. Polar coordinates and applications

Let's suppose that either the integrand or the region of integration comes out simpler in polar coordinates ($x = r\cos\theta$ and $y = r\sin\theta$). Let suppose we have a small change in r and θ. The small change Δr in r gives us two concentric circles and the small change $\Delta\theta$ in θ gives us an angular wedge.

If the changes are small, we almost get a rectangle with sides Δr and $r\Delta\theta$,

$$\Delta A \approx r\Delta r\Delta\theta.$$

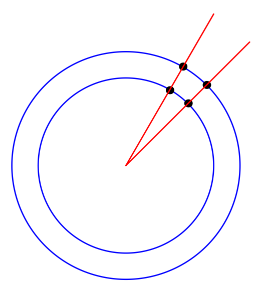

Figure 1. Small changes in r and θ

Taking the limit as Δr and $\Delta\theta$ go to zero, we get

$$\mathrm{d}A = r\mathrm{d}r\mathrm{d}\theta.$$

Example 16.1. *Compute the volume of $f(x,y) = x+3y$ over the circle $x^2+y^2 \leq 1$.*

$$\iint_R (x+3y)\,\mathrm{d}A = \int_0^{2\pi}\int_0^1 r^2(\cos\theta + 3\sin\theta)\,\mathrm{d}r\mathrm{d}\theta.$$

The inner integral is

$$\int_0^1 r^2(\cos\theta + 3\sin\theta)\,\mathrm{d}r = \left[\frac{r^3}{3}(\cos\theta + 3\sin\theta)\right]_0^1 = \frac{1}{3}\cos\theta + \sin\theta.$$

The outer integal is

$$\int_0^{2\pi} \frac{1}{3}\cos\theta + \sin\theta\,\mathrm{d}\theta = \left[\frac{1}{3}\sin\theta - \cos\theta\right]_0^{2\pi} = 0.$$

Example 16.2. *What is*

$$\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x?$$

Let

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x.$$

Then

$$\begin{aligned} I^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,\mathrm{d}y\right) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,\mathrm{d}x\mathrm{d}y \\ &= \iint_{\mathbb{R}^2} e^{-x^2-y^2}\,\mathrm{d}A \\ &= \int_0^{2\pi}\int_0^{\infty} re^{-r^2}\,\mathrm{d}r\mathrm{d}\theta. \end{aligned}$$

The inner integral is

$$\int_0^{\infty} re^{-r^2}\,\mathrm{d}r = \left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty} = \frac{1}{2}.$$

The outer integral is

$$\int_0^{2\pi} \frac{1}{2}\,\mathrm{d}\theta = \left[\frac{\theta}{2}\right]_0^{2\pi} = \pi.$$

So $I = \sqrt{\pi}$.

We can use double integrals to compute some interesting quantities. If we want to compute the area of a region, then just integrate 1,

$$\iint_R 1\,\mathrm{d}A.$$

The point is that the volume under the graph of a function of constant height is the area of the base times the height.

If we have a material whose mass density,

$$\delta(x, y) = \lim \frac{\Delta m}{\Delta A},$$

is a function of the position, then the total mass is

$$M = \iint_R \delta\,\mathrm{d}A.$$

Recall that the average of a function $f(x, y)$ over the region R is

$$\bar{f} = \frac{1}{A}\iint_R f\,\mathrm{d}A,$$

where A is the area of R. The centre of mass or centroid $(\bar{x}, \bar{y})$ of a plate with density δ is given by

$$\bar{x} = \frac{1}{M} \iint_R x\delta \, \mathrm{d}A \qquad \text{and} \qquad \bar{y} = \frac{1}{M} \iint_R y\delta \, \mathrm{d}A$$

The moment of inertia is a measure of how hard it is to rotate an object (in just the same way that the mass measures how hard it is to move an object). Suppose we have a mass m rotating in a circle of radius r with angular speed

$$\omega = \frac{d\theta}{dt}.$$

The velocity is $v = r\omega$. So the kinetic engergy is

$$\frac{1}{2}mv^2 = \frac{1}{2}mr^2\omega^2.$$

The moment of inertia is

$$I_0 = mr^2.$$

For a solid with density δ, the moment of inertia about the origin is

$$I_0 = \iint_R r^2\delta \, \mathrm{d}A.$$

The moment of inertia about the x-axis is

$$I_0 = \iint_R y^2\delta \, \mathrm{d}A.$$

Here y^2 is the square of the distance to the x-axis. In fact, one can think of the moment of inertia about the origin as the moment of inertia about the z-axis in $\mathbb{R}^3$. For a point in the plane, the square of the distance to the z-axis is r^2.

Example 16.3. *What is the moment of inertia of a disc of radius a about its centre?*

Here we assume that the density is one. Put the disc in the plane centred at the origin. Let R be the circle $x^2 + y^2 \leq a^2$. The moment of inertia is

$$\iint_R r^2 \, \mathrm{d}A = \int_0^{2\pi} \int_0^a r^3 \, \mathrm{d}r\mathrm{d}\theta.$$

The inner integral is

$$\int_0^a r^3 \, \mathrm{d}r = \left[\frac{r^4}{4}\right]_0^a = \frac{a^4}{4}.$$

The outer integral is

$$\int_0^{2\pi} \frac{a^4}{4}\,\mathrm{d}\theta = \left[\frac{a^4}{4}\theta\right]_0^{2\pi} = \frac{\pi a^4}{2}.$$

Now what happens if we try to compute the moment of inertia about a point of the circumference? Put the circle so its centre is at $(a, 0)$, so that the origin is the point on the circumference. Clearly we would like to use polar coordinates again.

The equation of the circle in Cartesian coordinates is

$$(x-a)^2 + y^2 = a^2.$$

Expanding we get

$$x^2 + y^2 = 2ax.$$

So

$$r^2 = 2ar\cos\theta,$$

that is

$$r = 2a\cos\theta.$$

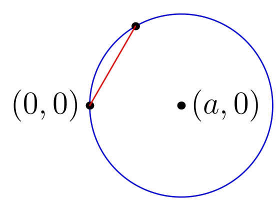

FIGURE 2. Limits of integration

The moment of inertia about the origin is therefore

$$I_0 = \iint_R r^2\,\mathrm{d}A = \int_0^{\pi}\int_0^{2a\cos\theta} r^3\,\mathrm{d}r\mathrm{d}\theta.$$

The inner integral is

$$\int_0^{2a\cos\theta} r^3\,\mathrm{d}r = \left[\frac{1}{4}r^4\right]_0^{2a\cos\theta} = 4a^4\cos^4\theta.$$

The outer integral is

$$\int_0^{\pi} 4a^4\cos^4\theta\,\mathrm{d}\theta = \frac{3}{2}\pi a^4.$$