15. Partial differential equations; double integrals

15.1. **Partial differential equations.** Recall that many functions of one variable are characterised by a(n ordinary) differential equation.

$$\frac{dy}{dx} = ky.$$

The solutions to this ODE are

$$y(x) = ae^{kx},$$

where a is a constant.

A partial differential equation is an equation satisfied by a function of several variables; a preposterously large number of problems in nature are described by PDEs.

Heat equation: The heat in a sheet of metal is described by $w(x, y, t)$ where

$$\frac{\partial w}{\partial t} = k\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right).$$

The term on the right has special significance:

Laplace equation: Let $u(x, y)$ be a function in the plane.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

This is a solution to the heat equation which is static in time.

Wave equation: Let $w(x, t)$ represent the displacement of a guitar string, as a function of time:

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}.$$

For example, consider $w(x, t) = \sin(\omega t + kx)$, where ω and k are constants.

Question 15.1. *For which values of ω and k is w a solution of the wave equation?*

We plug in the formula for w and see what we get:

$$\frac{\partial w}{\partial t} = \omega \cos(\omega t + kx),$$

so that

$$\frac{\partial^2 w}{\partial t^2} = -\omega^2 \sin(\omega t + kx).$$

On the other hand,

$$\frac{\partial w}{\partial x} = k \cos(\omega t + kx),$$

so that
$$\frac{\partial^2 w}{\partial x^2} = -k^2 \sin(\omega t + kx).$$
So, if $\omega^2 = c^2k^2$, we have a solution to the wave equation.

15.2. **Double integrals.** Suppose we have a region R in the plane and a function f defined on R. The double integral is the volume between the graph of f and the region R in the xy-plane:
$$\iint_R f(x, y)\,\mathrm{d}A.$$
As usual this is the signed volume. To compute this integral, imagine cutting the region R into small pieces R_i. Pick a point (x_i, y_i) belonging to each piece. Then the volume is approximately the sum
$$\sum_i f(x_i, y_i)\Delta A_i,$$
where ΔA_i is the area of the region R_i. Taking the limit, as the area of each piece goes to zero, we get the volume.

How do we compute the double integral? First we imagine dividing the region into small rectangles. The area of each rectangle is
$$\Delta A_i = \Delta x_i \Delta y_i.$$
Summing first over y and then x, we can compute the area by first integrating over y and then x.

Example 15.2. *Compute the volume of $f(x, y) = 2x + 3y - 1$ over the rectangle $1 \leq x \leq 3$, $1 \leq y \leq 2$.*

$$\iint_R (2x + 3y - 1)\,\mathrm{d}A = \int_{x=1}^{x=3} \int_{y=1}^{y=2} (2x + 3y - 1)\,\mathrm{d}y\mathrm{d}x.$$
First, we compute the inner integral
$$\int_{y=1}^{y=2} (2x+3y-1)\,\mathrm{d}y = \Big[2xy+3y^2/2-y\Big]_1^2 = (4x+6-2)-(2x+3/2-1) = 2x+7/2.$$
Now compute the outer integral
$$\int_{x=1}^{x=3} 2x + 9/2\,\mathrm{d}x = \Big[x^2 + 7x/2\Big]_1^3 = (9 + 21/2) - (1 + 7/2) = 15.$$

Note that we can switch the order of integration. If we first sum over x and then y, then we first integrate over x and then over y. Of course we get the same answer:

$$\iint_R (2x + 3y - 1)\,\mathrm{d}A = \int_{y=1}^{y=2} \int_{x=1}^{x=3} (2x + 3y - 1)\,\mathrm{d}x\mathrm{d}y.$$

Compute the inner integral

$$\int_{x=1}^{x=3} (2x+3y-1)\,\mathrm{d}x = \left[x^2+3yx-x\right]_1^3 = (9+9y-3)-(1+3y-1) = 6+6y.$$

Now compute the outer integral

$$\int_{y=1}^{y=2} 6 + 6y\,\mathrm{d}y = \left[6y + 3y^2\right]_1^2 = 12 + 12 - 6 - 3 = 15.$$

What happens when we want to integrate over a region which is not a rectangle?

Example 15.3. *Compute the volume of $f(x, y) = x+3y$ over the circle $x^2 + y^2 \leq 1$.*

The outer limits for x are relatively straightforward, the largest value of x is 1 and the smallest -1. But the limits for y are dependent on x. Imagine fixing a value for x; we get a vertical line. The lower limit for y represents where this line first meets the region. The upper limit for y represents where this line leaves the region.

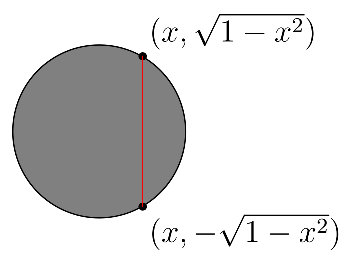

FIGURE 1. Limits for y

$$\iint_R (x + 3y)\,\mathrm{d}A = \int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} (x + 3y)\,\mathrm{d}y\mathrm{d}x.$$

Inner integral:

$$\int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} (2x + 3y)\,\mathrm{d}y = \left[xy + 3y^2/2\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} = 2x\sqrt{1 - x^2}.$$

Outer integral:

$$\int_{x=-1}^{x=1} 2x\sqrt{1-x^2}\,\mathrm{d}x = \Big[-2/3(1-x^2)^{3/2}\Big]_{-1}^{1} = 0.$$

Notice that in retrospect we could have predicted that the integral is zero. The function $f(x,y)$ is a sum of x and $3y$. x is odd with respect to x and y is odd with respect to y, so both pieces integrate to zero.

Example 15.4. *Compute*

$$\int_0^1 \int_x^1 e^{y^2}\,\mathrm{d}y\mathrm{d}x.$$

Note that we cannot compute the inner integral,

$$\int_x^1 e^{y^2}\,\mathrm{d}y.$$

since there is no way to integrate e^{y^2}. Instead, let's switch the order of integration. To do this, we first have to determine the region we are integrating over. x ranges from 0 to 1. For a given value of x, y ranges from x to 1. So the region of integration is the triangle $x \geq 0$, $y \leq 1$ and $y \geq x$.

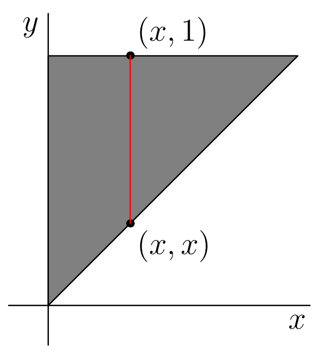

FIGURE 2. Region of integration

So we have

$$\int_0^1 \int_x^1 e^{y^2}\,\mathrm{d}y\mathrm{d}x = \int_0^1 \int_0^y e^{y^2}\,\mathrm{d}x\mathrm{d}y.$$

The inner integral is

$$\int_0^y e^{y^2}\,\mathrm{d}x = \Big[xe^{y^2}\Big]_0^y = ye^{y^2}.$$

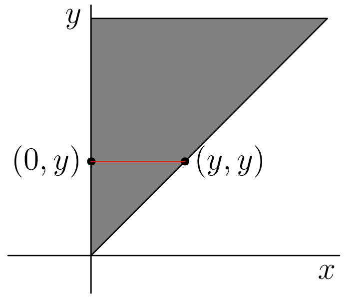

Figure 3. Limits of integral

The outer integral is

$$\int_0^1 ye^{y^2}\,\mathrm{d}y = \left[1/2e^{y^2}\right]_0^1 = 1/2(e-1).$$