10. Second derivative test

Let's turn to the problem of determining the nature of the critical points. Recall that there are three possibilities; either we have a local maximum, a local minimum or a saddle point.

Let's start with the key case, a quadratic polynomial.

$$f(x, y) = ax^2 + bxy + cy^2.$$

The basic trick is to complete the square. For example,

$$f(x, y) = x^2 - 2xy + 3y^2 = (x - y)^2 + 2y^2.$$

If we take the partials, we get

$$2ax + by \qquad \text{and} \qquad bx + 2cy.$$

Setting these equal to zero, we get

$$\begin{aligned} 2ax + by &= 0 \\ bx + 2cy &= 0. \end{aligned}$$

A homogeneous pair of linear equations. We can rewrite this as a matrix equation:

$$\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Provided

$$\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$$

is an invertible matrix (that is, the determinant $4ac - b^2 \neq 0$), this has the unique solution

$$(x, y) = (0, 0).$$

So we just want to know what sort of critical point we have at the origin. Let's suppose that $a \neq 0$, then

$$\begin{aligned} z &= a\left(x^2 + \frac{b}{a}xy\right) + cy^2 \\ &= a\left(x + \frac{b}{2a}y\right)^2 + \left(c - \frac{b^2}{4a}\right)y^2 \\ &= \frac{1}{4a}\left(4a^2\left(x + \frac{b}{2a}y\right)^2 + \left(4ac - b^2\right)y^2\right). \end{aligned}$$

To get from the first line to the second line we completed the square. The advantage of the third line is that we know that $4a^2$ is always positive (we are assuming that $a \neq 0$).

There are three cases.

Case 1: $a > 0$, $4ac - b^2 > 0$. $f(x, y)$ is a sum of two squares and $(0, 0)$ is a local minimum.

Case 2: $a < 0$, $4ac - b^2 > 0$. $-f(x, y)$ is a sum of two squares and $(0, 0)$ is a local maximum.

Case 3: $4ac - b^2 < 0$. $f(x, y)$ is the difference of two squares and $f(x, y)$ is a saddle point.

The case $4ac-b^2 = 0$ is a degenerate case (the second derivative test fails).

For the second derivative test, one looks at the second derivatives of f. There are four second derivatives,

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy} \qquad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}.$$

Example 10.1. *Let $f(x, y) = x^3y + xy$.*

We have

$$f_x = 3x^2y + y \qquad \text{and} \qquad f_y = x^3 + x$$

$$f_{xx} = 6xy \qquad f_{yy} = 0$$
$$f_{xy} = 3x^2 + 1 \qquad f_{yx} = 3x^2 + 1.$$

In general, $f_{xy} = f_{yx}$ (except for some pathological functions). Suppose that (x_0, y_0) is a critical point of $f(x, y)$. Let $A = f_{xx}(x_0, y_0)$, $B = f_{xy}(x_0, y_0)$ and $C = f_{yy}(x_0, y_0)$.

If $AC - B^2 > 0$ then we have a local maximum or minimum; a minimum if $A > 0$ and a maximum if $A < 0$.

If $AC - B^2 < 0$ then we have a saddle point.

If $AC - B^2 = 0$ then the second derivative test is inconclusive (a nice way to say that we know absolutely nothing and we were wasting our time needlessly computing 2nd derivatives).

Let's see what happens if

$$f(x, y) = ax^2 + bxy + cy^2.$$

Then

$$A = f_{xx}(0, 0) = 2a \qquad B = f_{xy}(0, 0) = b \qquad \text{and} \qquad C = f_{yy}(0, 0) = 2c.$$

So

$$AC - B^2 = 4ac - b^2,$$

and the test works.

In general, we have

$$\Delta f \approx f_x(p)(x-x_0)+f_y(y-y_0)+\frac{1}{2}f_{xx}(p)(x-x_0)^2+f_{xy}(p)(x-x_0)(y-y_0)+\frac{1}{2}f_{yy}(p)(y-y_0)^2,$$

where the partials are all computed at $p = (x_0, y_0)$. At a critical point we have

$$\Delta f \approx \frac{A}{2}(x - x_0)^2 + B(x - x_0)(y - y_0) + \frac{C}{2}(y - y_0)^2.$$

Now if we want to find global maximum or minimum, we also have to worry about what happens on the boundary or as we approach infinity.

Example 10.2. *What are the maxima and minima of*

$$f(x, y) = x + y + \frac{1}{xy}?$$

First find critical points

$$f_x = 1 - \frac{1}{x^2y} \qquad \text{and} \qquad f_y = 1 - \frac{1}{xy^2}.$$

Setting these equal to zero, we get

$$x^2y = 1 \qquad \text{and} \qquad xy^2 = 1.$$

Dividing we get $x = y$. Hence $x^3 = 1$ and so $x = y = 1$. $(1, 1)$ is the only critical point.

We now apply the second derivative test.

$$f_{xx} = \frac{2}{x^3y} \qquad f_{xy} = \frac{1}{x^2y^2} \qquad \text{and} \qquad f_{yy} = \frac{2}{xy^3}.$$

So

$$A = f_{xx}(1, 1) = 2 \qquad B = f_{xy}(1, 1) = 1 \qquad \text{and} \qquad C = f_{yy}(1, 1) = 2.$$

$AC - B^2 = 4 - 1 = 3 > 0$, so we have either a local maximum or a local minimum. $A = 2 > 0$ so we have a local minimum. But what happens at the boundary and as we go to infinity? As $x \to 0$ $f(x, y) \to \infty$. Similarly if $y \to 0$, $x \to \infty$ or $y \to \infty$.

So $f(x, y)$ has a global minimum at $(x, y) = (1, 1)$ and there is no maximum.