MODEL ANSWERS TO HWK #9

1. Suppose that X has genus g. Let $D = (g+1)P$. By Riemann-Roch,

$$h^0(X, \mathcal{O}_X((g+1)P)) \geq (g+1) - g + 1 = 2.$$

Elements of $H^0(X, \mathcal{O}_X((g+1)P))$ correspond to rational functions with pole no worse than $(g+1)P$. As the space of regular functions, which is the same as the space of constant functions, spans a one dimensional subspace, we may find a non-constant rational function with poles only at P.

2. By (1) we may find $f_1, f_2, \dots, f_r$ where f_i has exactly one pole at P_i. Let

$$f = \sum a_i f_i,$$

where a_i are scalars. Then f only has poles along $P_1, P_2, \dots, P_r$. The set of points $(a_1, a_2, \dots, a_r) \in \mathbb{A}^r$ where f has a pole at P_i is an open subset U_i, which is non-empty since f_i defines a point of U_i. Since the groundfield is algebraically closed, it is infinite and $U = \cap U_i$ is non-empty. But then if $(a_1, a_2, \dots, a_r) \in U$ then f has a pole at $P_1, P_2, \dots, P_r$.

3. X is a quasi-projective variety so we may embed it into a projective variety $\bar{X}$. Replacing $\bar{X}$ by its normalisation, which does not change X as X is normal, we may assume that $\bar{X}$ is a curve.

Let $P_1, P_2, \dots, P_r$ be the points belonging to $\bar{X}$ but not to X. By (2) we may find a rational function whose polar locus is precisely X. f determines a morphism

$$\bar{X} \longrightarrow \mathbb{P}^1,$$

such that the inverse image of ∞ is $P_1, P_2, \dots, P_r$. The restriction of this morphism to X defines a finite morphism to $\mathbb{A}^1$. But then X is affine by (III.4.2).

4. By (III.3.1) we may assume that X is reduced. By (III.3.2) we may assume that X is irreducible, that is, we may assume that X is integral. But then X is affine by 3.

5. Let $d = \deg D \geq 0$. Note that

$$h^0(C, \mathcal{O}_X(K_C - D)) \leq h^0(C, \mathcal{O}_C(K_C)) = g.$$

Hence, by Riemann-Roch,

$$h^0(X, \mathcal{O}_X(D)) = d - g + 1 + h^0(C, \mathcal{O}_X(K_C - D)) \leq d + 1.$$

Thus

$$|D| \leq \deg D.$$

Clearly we have equality if $D = 0$ or $g = 0$. Suppose that $D \neq 0$ and we have equality. By assumption

$$D = \sum_{i=1}^{d} P_i.$$

Let $P = P_d$ and consider the exact sequence

$$0 \longrightarrow \mathcal{O}_C(D - P) \longrightarrow \mathcal{O}_C(D) \longrightarrow \mathcal{O}_P(D) \longrightarrow 0$$

Taking cohomology we see that

$$\deg D - 1 \leq |D| - 1 \leq |D - P| \leq \deg D - 1,$$

so that

$$|D - P| = \deg(D - P).$$

By induction on the degree we see that $|P_1| = \deg P_1 = 1$ so that there is a morphism of degree one to $\mathbb{P}^1$. But this morphism is then an isomorphism, that is, $C = \mathbb{P}^1$ and $g = 0$.

6. Let D be a divisor of degree $g + 1$. Then

$$h^0(C, \mathcal{O}_C(D)) \geq g + 1 - g + 1 = 2.$$

Therefore we may find a non-constant rational function f with pole no worse than D. The corresponding morphism

$$X \longrightarrow \mathbb{P}^1$$

has degree equal to the degree of the polar locus of f, which is at most $g + 1$.