MODEL ANSWERS TO HWK #8

5.5 We prove (a) and (c) by induction on the codimension of Y. By assumption, Y is the intersection of a hypersurface of degree d and another complete intersection subvariety Z of codimension one less than the codimension of Y. There is an exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

By assumption $\mathcal{I} = \mathcal{O}_Z(-d)$. Twisting by $\mathcal{O}_Z(n)$ preserves exactness and by induction we have

$$h^i(Z, \mathcal{O}_Z(m)) = 0,$$

for all $0 < i < q+1$ and all positive integers m. This gives (c) and we have

$$H^0(Z, \mathcal{O}_Z(n)) \longrightarrow H^0(Y, \mathcal{O}_Y(n)),$$

is surjective. Composing this gives (a).
(b) Note that $h^0(X, \mathcal{O}_X) = 1$ and $h^0(Y, \mathcal{O}_Y)$ is the number of connected components of Y. Take $n = 0$.
(d) Immediate from (c).
5.6 (a)
(1) There is an exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Q \longrightarrow 0.$$

Note that $\mathcal{I} = \mathcal{O}_X(-2)$. Twisting by $\mathcal{O}_X(n)$, we get that

$$h^1(Q, \mathcal{O}_Q(n, n)) = 0.$$

There is an exact sequence

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{O}_Q \longrightarrow \mathcal{O}_F \longrightarrow 0,$$

where F is a curve of type $(1, 0)$, that is a curve of the form $\{p\} \times \mathbb{P}^1$. We have $\mathcal{J} = \mathcal{O}_Q(-1, 0)$. Twisting by $\mathcal{O}_Q(n, n)$ we have

$$0 \longrightarrow \mathcal{O}_Q(n-1, n) \longrightarrow \mathcal{O}_Q(n, n) \longrightarrow \mathcal{O}_F(n) \longrightarrow 0,$$

Now we have already seen that $h^1(Q, \mathcal{O}_Q(n, n)) = 0$ and the map

$$H^0(Q, \mathcal{O}_Q(n, n)) \longrightarrow H^0(F, \mathcal{O}_F(n)),$$

is surjective by inspection. It follows that

$$h^1(Q, \mathcal{O}_Q(n-1, n)) = 0.$$

(2) If $a \neq b$ then we might as well assume that $a < b$. Twisting the second exact sequence above by $(a+1, b)$, we get an exact sequence

$$0 \longrightarrow \mathcal{O}_Q(a,b) \longrightarrow \mathcal{O}_Q(a+1,b) \longrightarrow \mathcal{O}_F(b) \longrightarrow 0,$$

As $b < 0$, $h^0(F, \mathcal{O}_F(b)) = 0$ and so taking the long exact sequence of cohomology, we get

$$h^1(Q, \mathcal{O}_Q(a,b)) \leq h^1(Q, \mathcal{O}_Q(a+1,b)).$$

Thus we may reduce to the case $a = b$, in which case we can apply (1).
(3) Let Y be a curve of type a, the union of a copies of $\mathbb{P}^1$. Then there is an exact sequence

$$0 \longrightarrow \mathcal{O}_Q(a,0) \longrightarrow \mathcal{O}_Q \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

As

$$h^0(Y, \mathcal{O}_Y) = a \qquad \text{and} \qquad h^0(Q, \mathcal{O}_Q) = 1,$$

the result is clear.
(b)
(1) There is an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}_Q \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

As Y is a curve of type (a,b), we have $\mathcal{K} = \mathcal{O}_Q(-a,-b)$. (a.2) implies that

$$H^0(Q, \mathcal{O}_Q) \longrightarrow H^0(Y, \mathcal{O}_Y),$$

is surjective, so that

$$h^0(Y, \mathcal{O}_Y) \leq 1.$$

But the LHS is equal to the number of connected components of Y.
(2) Follows from Bertini and the fact that Y is connected.
(3) By (II.5.1.4), Y is projectively normal if and only if

$$H^0(X, \mathcal{O}_X(n)) \longrightarrow H^0(Y, \mathcal{O}_Y(n)),$$

is surjective for all integers n. Since

$$H^0(X, \mathcal{O}_X(n)) \longrightarrow H^0(Q, \mathcal{O}_Q(n)),$$

is surjective, Y is projectively normal if and only if

$$H^0(Q, \mathcal{O}_Q(n)) \longrightarrow H^0(Y, \mathcal{O}_Y(n)),$$

is surjective. As $h^1(Q, \mathcal{O}_Q(n)) = 0$, this holds if and only if

$$h^1(Q, \mathcal{O}_Q(n-a, n-b)) = 0,$$

for all integers n.
If $|a-b| \leq 1$, then Y is projectively normal by (a.1). Otherwise if $a < b-1$ and we take $n = b$, then (a.3) shows that Y is not projectively normal.

(c) As always, consider the usual exact sequence

$$0 \longrightarrow \mathcal{O}_Q(n-a, n-b) \longrightarrow \mathcal{O}_Q(n) \longrightarrow \mathcal{O}_Y(n) \longrightarrow 0.$$

If n is sufficiently large, then by Serre vanishing, we have that

$$\chi(Y, \mathcal{O}_Y(n)) = h^0(Y, \mathcal{O}_Y(n)) = h^0(Q, \mathcal{O}_Q(n)) - h^0(Q, \mathcal{O}_Q(n-a, n-b)).$$

Suppose $a \neq b$. Then we may suppose that $a < b$. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_Q(n-a-1, n-b) \longrightarrow \mathcal{O}_Q(n-a, n-b) \longrightarrow \mathcal{O}_F(n-b) \longrightarrow 0.$$

Assuming n is sufficiently large, we get that

$$h^0(Q, \mathcal{O}_Q(n-a, n-b)) = h^0(Q, \mathcal{O}_Q(n-a-1, n-b)) + (n-b+1),$$

so that

$$h^0(Q, \mathcal{O}_Q(n-a, n-b)) = h^0(Q, \mathcal{O}_Q(n-b, n-b)) + (b-a)(n-b+1).$$

On the other hand,

$$\begin{aligned} h^0(Q, \mathcal{O}_Q(n)) &= h^0(X, \mathcal{O}_X(n)) - h^0(X, \mathcal{O}_X(n-2)) \\ &= \binom{n+3}{3} - \binom{n+1}{3} \\ &= \frac{(n+1)[(n+3)(n+2) - n(n-1)]}{6} \\ &= (n+1)^2. \end{aligned}$$

So

$$\begin{aligned} \chi(Y, \mathcal{O}_Y(n)) &= (n+1)^2 - (n+1-b)^2 - (b-a)(n-b+1) \\ &= 2b(n+1) - b^2 - bn + an + b^2 - ab - b + a \\ &= (a+b)n + a + b - ab, \end{aligned}$$

which is then the Hilbert polynomial of Y. It follows that $p_a(Y) = ab - a - b + 1$.

5.8 (a) Apply (II.6.7).

(b) As $\mathcal{L}$ is very ample, there is an embedding of $\tilde{X}$ into $\mathbb{P}^n$ such that $\mathcal{L} = \mathcal{O}_{\tilde{X}}(1)$. Let H be a hyperplane section which avoids the inverse image of the singular locus of X. Then $D = H \cap \tilde{X}$ is a divisor on $\tilde{X}$ such that $D = \sum P_i$, where $Q_i = f(P_i)$ is a smooth point of X, for each i. Let $E = \sum Q_i$. Then E is a Cartier divisor on X. Let $\mathcal{L}_0 = \mathcal{O}_X(E)$. Then $f^*\mathcal{L}_0 \simeq \mathcal{L}$. By (5.7.d), $\mathcal{L}_0$ is ample. By (II.7.6) some power of $\mathcal{L}$ is very ample and it follows by (II.5.16.1) that X is projective.

(c) Let X' be the disjoint union of the $X_1, X_2, \dots, X_r$. Then there is a natural morphism $i\colon X' \longrightarrow X$. This gives rise to an exact sequence

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow i_*\mathcal{O}_{X'}^* \longrightarrow \delta \longrightarrow 0,$$

where δ is supported on a zero dimensional scheme. Taking the long exact sequence of cohomology and using (4.5), we see that

$$\operatorname{Pic} X \longrightarrow \bigoplus \operatorname{Pic} X_i,$$

is surjective. Pick ample line bundles $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_r$ on $X_1, X_2, \dots, X_r$. Then we may find a line bundle $\mathcal{L}$ such that $\mathcal{L}|_{X_i} \simeq \mathcal{L}_i$. $\mathcal{L}$ is ample by (5.7.c) and so X is projective.
(d) Let $\mathcal{J}$ be the ideal sheaf of X_{red} in X. Let $\mathcal{X}_k$ be subscheme of X determined by $\mathcal{I}^k$. Then $X_k \subset X_{k+1}$ and the ideal sheaf $\mathcal{I}_k$ squares to zero. By (4.6), there is an exact sequence

$$\operatorname{Pic} X_{k+1} \longrightarrow \operatorname{Pic} X_k \longrightarrow H^2(X, \mathcal{I}_k).$$

By (2.7) the last group is zero. Composing, we get that

$$\operatorname{Pic} X \longrightarrow \operatorname{Pic} X_{\text{red}},$$

is surjective. Let $\mathcal{M}$ be an ample line bundle on X_{red} and let $\mathcal{L}$ be a line bundle on X whose restriction to X_{red} is $\mathcal{M}$. (5.7.c) implies that $\mathcal{L}$ is ample.
5.9. Note that

$$\frac{x_j}{x_i}\,\mathrm{d}\left(\frac{x_i}{x_j}\right) = \frac{1}{x_i x_j}(x_j\,\mathrm{d}x_i - x_i\,\mathrm{d}x_j) = \frac{\mathrm{d}x_i}{x_i} - \frac{\mathrm{d}x_j}{x_j}.$$

It follows that we do have a 1-cocycle.
As in the hint, we just have to show that $\delta(\mathcal{O}_X(1)) \neq 0$. Note that

$$\mathcal{L}|_{U_i} \simeq \mathcal{O}_{U_i},$$

via the map which sends $f \in \mathcal{O}_{U_i}(V)$ to $x_i f$, so that $\mathcal{O}_X(1)$ is represented by the 1-cocycle α

$$\frac{x_1}{x_0} \quad \text{on } U_{01}, \qquad \frac{x_2}{x_1} \quad \text{on } U_{12}, \text{ and} \qquad \frac{x_0}{x_2} \quad \text{on } U_{20}.$$

Let $\mathcal{U} = \{U_0, U_1, U_2\}$. The relevant commutative diagram is then

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^1(\mathcal{U}, \omega) & \longrightarrow & C^1(\mathcal{U}, \mathcal{O}_{X'}^*) & \longrightarrow & C^1(\mathcal{U}, \mathcal{O}_X^*) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^2(\mathcal{U}, \omega) & \longrightarrow & C^2(\mathcal{U}, \mathcal{O}_{X'}^*) & \longrightarrow & C^2(\mathcal{U}, \mathcal{O}_X^*) & \longrightarrow & 0.
\end{array}$$

Now

$$\Gamma(U_{ij}, \mathcal{O}_{\mathbb{P}^2}^*) = K[\frac{x_i}{x_j}, \frac{x_j}{x_i}].$$

On the other hand,

$$\Gamma(U_{ij}, \mathcal{O}_{X'}) \simeq \Gamma(U_{ij}, \mathcal{O}_{\mathbb{P}^2})[\epsilon_{ij}] \simeq K[\frac{x_i}{x_j}, \frac{x_j}{x_i}, \frac{x_k}{x_i}][\epsilon_{ij}],$$

where $\epsilon_{ij}^2 = 0$ and $\{ i, j, k \} = \{ 1, 2, 3 \}$. Note that the units of any ring with an element ϵ such that $\epsilon^2 = 0$ are of the form $a + \epsilon b$, where a is a unit.

ξ determines a derivation on U_{ij} given by a map

$$\Gamma(U_{ij}, \mathcal{O}_{\mathbb{P}^2}) \longrightarrow \Gamma(U_{ij}, \omega).$$

Note that if E is a two dimensional vector space, the identification of $\mathrm{Hom}(E, \wedge^2 E)$ with E is via the map which sends a vector $v \in E$ to the linear map

$$w \longrightarrow w \wedge v.$$

Now

$$\mathrm{d}\left(\frac{x_i}{x_j}\right) \wedge \xi_{ij} = \frac{x_i}{x_j}\xi_{ij} \wedge \xi_{ij} = 0.$$

Similarly

$$\mathrm{d}\left(\frac{x_j}{x_i}\right) \wedge \xi_{ij} = -\frac{x_j}{x_i}\xi_{ji} \wedge \xi_{ji} = 0.$$

But

$$\mathrm{d}\left(\frac{x_k}{x_j}\right) \wedge \xi_{ij} = \frac{x_j}{x_i}\mathrm{d}\left(\frac{x_k}{x_j}\right) \wedge \mathrm{d}\left(\frac{x_i}{x_j}\right).$$

The derivation corresponding to ξ_{ij} is therefore

$$\frac{x_i}{x_j} \longrightarrow 0, \qquad \frac{x_j}{x_i} \longrightarrow 0 \qquad \text{and} \qquad \frac{x_k}{x_j} \longrightarrow \frac{x_j}{x_i}\mathrm{d}\left(\frac{x_k}{x_j}\right) \wedge \mathrm{d}\left(\frac{x_i}{x_j}\right).$$

We can lift the 1-cocycle α to the 1-cochain $\beta = (\beta_2, \beta_0, \beta_1)$ in $C^1(\mathcal{U}, \mathcal{O}^*_{X'})$, This maps to the 2-cocycle γ in $C^2(\mathcal{U}, \mathcal{O}^*_{X'}) = \Gamma(U_{012}, \mathcal{O}^*_{X'})$. To figure out the image of β, let's calculate everything in terms of the isomorphism

$$\phi\colon \Gamma(U_0, \mathcal{O}_{X'}) \longrightarrow K[\frac{x_1}{x_0}, \frac{x_2}{x_0}][\epsilon].$$

We have

$$\begin{aligned}
\phi(\beta_2) &= \frac{x_1}{x_0} \\
\phi(\beta_0) &= \frac{x_2}{x_1} + \epsilon\frac{x_0}{x_1}\mathrm{d}\left(\frac{x_2}{x_0}\right) \wedge \mathrm{d}\left(\frac{x_1}{x_0}\right) \\
\phi(\beta_1) &= \frac{x_0}{x_2}.
\end{aligned}$$

Thus γ is represented by

$$\frac{x_1}{x_0}\left(\frac{x_2}{x_1}+\epsilon\frac{x_0}{x_1}\mathrm{d}\left(\frac{x_2}{x_0}\right)\wedge\mathrm{d}\left(\frac{x_1}{x_0}\right)\right)\frac{x_0}{x_2}=1+\epsilon\frac{x_0}{x_2}\mathrm{d}\left(\frac{x_2}{x_0}\right)\wedge\mathrm{d}\left(\frac{x_1}{x_0}\right)$$

Thus γ lifts to the element

$$\frac{x_0}{x_2}\mathrm{d}\left(\frac{x_2}{x_0}\right)\wedge\mathrm{d}\left(\frac{x_1}{x_0}\right)$$

of $C^2(\mathcal{U},\omega)$, which is a non-zero element of $H^2(X,\omega)$.

2. Note that $e=0$ if and only if $1\in H^0(X,\mathcal{O}_X)$ is in the image of

$$\phi\colon H^0(X,\mathcal{G})\longrightarrow H^0(X,\mathcal{O}_X).$$

Suppose that $\phi(\sigma)=1$. Define a sheaf homomorphism

$$\mathcal{O}_X\longrightarrow\mathcal{G},$$

by sending $f\in\mathcal{G}(U)$ to $f\sigma|_U$. It is easy to see that this defines a splitting.

Conversely, if the exact sequence is split, it is clear that ϕ is surjective.

3. Let $X=\mathbb{P}^1$. Pick m such that $\mathcal{E}(m)$ is globally generated. Then we get a morphism of sheaves

$$\mathcal{O}_X\longrightarrow\mathcal{E}(m).$$

If we dualise this morphism we get a morphism

$$\mathcal{E}^*(-m)\longrightarrow\mathcal{O}_X.$$

Let $\mathcal{I}$ be the image. Then $\mathcal{I}$ is a $\mathcal{O}_X$-submodule of $\mathcal{O}_X$, that is, $\mathcal{I}$ is an ideal sheaf. As $X=\mathbb{P}^1$ is a curve, this defines a zero dimensional subscheme. Let $D\geq 0$ be the associated divisor. Then $\mathcal{I}=\mathcal{I}_z=\mathcal{O}_X(-D)=\mathcal{O}_X(-d)$ where $d=\deg D$. So we get a surjective morphism

$$\mathcal{E}^*(-m)\longrightarrow\mathcal{O}_X(-d).$$

Twisting by d we get a surjective morphism

$$\mathcal{E}^*(-m+d)\longrightarrow\mathcal{O}_X.$$

Finally replacing $m-d$ by m we have a surjective morphism

$$\mathcal{E}^*(-m)\longrightarrow\mathcal{O}_X.$$

Let $\mathcal{K}$ be the kernel. If we pick a point $p\in X$, we get an exact sequence on stalks,

$$0\longrightarrow\mathcal{K}_p\longrightarrow\mathcal{O}^r_{X,p}\longrightarrow\mathcal{O}_{X,p}\longrightarrow 0,$$

where

$$\mathcal{O}_{X,p}\simeq k[x]_x.$$

As $k[x]_x$ is a PID, it follows that

$$\mathcal{K}_p \simeq \mathcal{O}_{X,p}^{r-1},$$

so that $\mathcal{K}$ is locally free of rank $r-1$. Denoting by $\mathcal{Q}$ the dual of $\mathcal{K}$, we get an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E}(m) \longrightarrow \mathcal{Q} \longrightarrow 0.$$

As $\mathcal{Q}$ is locally free of rank $r-1$, by induction on r, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E}(m) \longrightarrow \bigoplus_{i=1}^{r-1} \mathcal{O}_X(a_i) \longrightarrow 0,$$

for some integers $a_1, a_2, \dots, a_{r-1}$.
Suppose that we tensor this exact sequence by $\mathcal{O}_X(-k)$, where

$$k > \max_i a_i,$$

and $k > 0$. Then $h^0(X, \mathcal{E}(m-k)) = 0$, since both $h^0(X, \mathcal{O}_X(-k)) = 0$ and $h^0(X, \mathcal{Q}(-k)) = 0$. It follows that there is a smallest integer m' such that $h^0(X, \mathcal{E}(m')) \neq 0$. Note if we go through the process above to obtain an exact sequence, then the corresponding ideal sheaf $\mathcal{I}'$ is trivial, that is, $d' = 0$, by minimality of m'. Therefore, replacing m by m', we get the same exact sequence as before, but in addition we also have $h^0(X, \mathcal{E}(m-1)) = 0$.
As

$$h^1(X, \mathcal{O}_X(-1)) = h^0(X, \omega_X(1)) = h^0(X, \mathcal{O}_X(-1)) = 0,$$

it follows that

$$h^0(X, \mathcal{Q}(-1)) = 0.$$

In particular $a_i - 1 < 0$, that is $a_i < 1$. It follows that

$$h^1(X, \mathcal{K}) = \sum_i h^1(X, \mathcal{O}_X(-a_i)) = \sum_i h^0(X, \mathcal{O}_X(a_i - 2)) = 0.$$

But then (2) implies that the short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{E}^*(-m) \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

is split, so that $\mathcal{E}^*(-m)$ is a direct sum of line bundles. But then $\mathcal{E}(m)$ is a direct sum of the dual line bundles and so $\mathcal{E}$ is also a direct sum of line bundles.
It is easy to see that one can recover the sequence

$$a_1, a_2, \dots, a_r,$$

from the data of the

$$h^0(X, \mathcal{E}(d)),$$

for all integers d.