MODEL ANSWERS TO HWK #7

4.1 Let $\mathcal{U} = \{U_i\}$ be an open affine cover of X which is locally finite. Then $\mathcal{V} = \{V_i\}$ is an open affine cover of Y which is locally finite, where $V_i = f^{-1}(U_i)$. Note that, if I is a finite set of indices, then there are natural isomorphisms

$$H^0(U_I, \mathcal{F}) = H^0(V_I, f_*\mathcal{F}).$$

It follows that

$$H^*(\mathcal{U}, \mathcal{F}) \simeq H^*(\mathcal{V}, f_*\mathcal{F}).$$

But, by (II.5.8) we have that $f_*\mathcal{F}$ is quasi-coherent and so

$$H^*(X, \mathcal{F}) \simeq H^*(\mathcal{U}, \mathcal{F}) \qquad \text{and} \qquad H^*(Y, \mathcal{F}) \simeq H^*(\mathcal{V}, f_*\mathcal{F}).$$

4.2 (a) Let M and L be the functions fields of X and Y (that is, the residue fields of the generic points of X and Y). Then M/L is a finite field extension. Pick a basis $m_1, m_2, \dots, m_r$ of the L-vector space M. Let $U = \operatorname{Spec} A \subset X$ be an open affine subset of X. Then M is the field of fractions of A. We may find $a_1, a_2, \dots, a_r \in A$ such that $M = K(a_1, a_2, \dots, a_r)$. Define a morphism of sheaves

$$\mathcal{O}_X{}^r \longrightarrow \mathcal{K},$$

by sending $(f_1, f_2, \dots, f_r)$ to $\sum a_i f_i$. Let $\mathcal{M}$ be the image. Then $\mathcal{M}$ is a coherent sheaf as it is the image of a coherent sheaf. Almost by definition there is a morphism of sheaves

$$\mathcal{O}_Y f_*\mathcal{O}_X \longrightarrow$$

Taking the direct sum, pushing forward the map above and composing we get a morphism of $\mathcal{O}_Y$-modules,

$$\alpha\colon \mathcal{O}_Y^r \longrightarrow f_*\mathcal{M},$$

which is an isomorphism at the generic point, since then it reduces to the vector space isomorphism,

$$L^r \longrightarrow M.$$

(b) If we apply $\mathcal{H}\text{om}(\cdot, \mathcal{F})$ to α, we get a morphism of sheaves

$$\mathcal{H}\text{om}(f_*\mathcal{M}, \mathcal{F}) \longrightarrow \mathcal{F}^r,$$

which is certainly an isomorphism at the generic point. Note that

$$\mathcal{H}\text{om}(f_*\mathcal{M}, \mathcal{F}),$$

is a coherent $\mathcal{A} = f_*\mathcal{O}_X$-module. By (5.17e), there is a coherent $\mathcal{O}_X$-module $\mathcal{G}$ such that

$$f_*\mathcal{G} = \mathcal{H}\mathrm{om}(f_*\mathcal{M}, \mathcal{F}).$$

(c) Let $Y' \subset Y$ be a closed subscheme, let $X' \subset f^{-1}(Y')$ be a closed subset of the inverse such that the induced morphism $f'\colon X' \longrightarrow Y'$ is surjective. Note that $X' \subset X$ is affine, and f' is a finite morphism. Indeed, to check f' is finite, we may assume that $Y = \operatorname{Spec} B$ is affine and by assumption A is a finitely generated B-module. If I and J are the defining ideals of X' and Y', then $X' = \operatorname{Spec} A/I$, $Y' = \operatorname{Spec} B/J$ and it is clear that A/I is a finitely generated B/J-module.
Note that $f_{\mathrm{red}}\colon X_{\mathrm{red}} \longrightarrow Y_{\mathrm{red}}$ is a surjective finite morphism of noetherian, separated and reduced schemes. As (3.1) implies that Y is affine if and only if Y_{red} is affine, we may assume that X and Y are reduced.
Suppose that $Y' \subset Y$ is an irreducible component of Y. As f is surjective, there is an irreducible component X' of X which surjects to Y'. The induced morphism $f'\colon X' \longrightarrow Y$ is a surjective finite morphism of noetherian, separated and integral schemes. As (3.2) implies that Y is affine if and only if each irreducible component Y' is affine, we may assume that X and Y are integral. Let $\mathcal{F}$ be a quasi-coherent sheaf on Y. We check that

$$H^i(Y, \mathcal{F}) = 0,$$

for all $i > 0$. By Noetherian induction and (3.7), we may suppose that

$$H^i(Y', \mathcal{G}) = 0,$$

for all proper closed subsets and all quasi-coherent sheaves $\mathcal{G}$.
By (b), we may find an exact sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow f_*\mathcal{G} \longrightarrow \mathcal{F}^r \longrightarrow \mathcal{Q} \longrightarrow 0,$$

where $\mathcal{R}$ and $\mathcal{Q}$ are quasi-coherent sheaves, supported on proper closed subsets of Y. By induction,

$$H^i(Y, \mathcal{F}^r) = H^i(Y, f_*\mathcal{G}),$$

and the last group is isomorphic to

$$H^i(X, \mathcal{G}),$$

by (4.1). But this vanishes as X is affine and $\mathcal{G}$ is quasi-coherent. Thus

$$H^i(Y, \mathcal{F}) = 0,$$

for all $i > 0$ and all quasi-coherent sheaves $\mathcal{F}$, and so Y is affine by (3.7).
4.3 Let $\mathcal{U} = \{U_x, U_y\}$, where U_x is the complement of the x-axis and U_y is the complement of the y-axis. Then U_x and U_y are both isomorphic

to $\mathbb{A}^1 \times (\mathbb{A}^1 - \{0\})$, so that they are both affine. The intersection of U_x and U_y is $(\mathbb{A}^1 - \{0\}) \times (\mathbb{A}^1 - \{0\})$, which is again affine. As $\mathcal{O}_X$ is coherent, we have an isomorphism,

$$H^1(\mathcal{U}, \mathcal{O}_X) \simeq H^1(X, \mathcal{O}_X).$$

Now an element of $C^1(\mathcal{U}, \mathcal{O}_X)$ is nothing but a section of $H^0(U_x \cap U_y, \mathcal{O}_X)$. Since there are no triple intersections, every cochain is automatically a cocycle, so that

$$Z^1(\mathcal{U}, \mathcal{O}_X) = C^1(\mathcal{U}, \mathcal{O}_X) = k[x, y]_{xy}.$$

Now

$$C^0(\mathcal{U}, \mathcal{O}_X) = H^0(U_x, \mathcal{O}_X) \oplus H^0(U_y, \mathcal{O}_X).$$

Note that

$$H^0(U_x, \mathcal{O}_X) = k[x, y]_x \qquad \text{and} \qquad H^0(U_y, \mathcal{O}_X) = k[x, y]_y.$$

Thus

$$B^1(\mathcal{U}, \mathcal{O}_X) = k[x, y]_x + k[x, y]_y.$$

It follows that a basis of

$$H^1(X, \mathcal{O}_X),$$

is given by monomials of the form $x^i y^j$, where $i < 0$ and $j < 0$. In particular,

$$h^1(X, \mathcal{O}_X),$$

is not finite.

It is also interesting to calculate $H^1(X, \mathcal{O}_X)$ using the fact that X is toric. The fan F corresponding to X is the union of the two one dimensional cones spanned by e_1 and e_2 (but not including the cone spanned by e_1 and e_2) and the origin (which is a face of both one dimensional cones). Then the support of the fan F is

$$|F| = \{\, (x, 0) \,|\, x \geq 0 \,\} \cup \{\, (0, y) \,|\, y \geq 0 \,\}.$$

The 0 divisor is T-Cartier and corresponds to the zero function on F. According to (15.6),

$$H^1(X, \mathcal{O}_X),$$

decomposes as a direct sum of eigenspaces, indexed by $u \in M$, where each piece is given by a local cohomology group,

$$H^1_{Z(u)}(|F|, \mathbb{C}).$$

The last group is isomorphic to the relative cohomology of the pair

$$H^1(|F|, Z(u), \mathbb{C}).$$

The long exact sequence for the pair $Z(u) \subset |F|$ is:

$$\begin{aligned} 0 \longrightarrow H^0(|F|, |F| - Z(u), \mathbb{C}) \longrightarrow H^0(|F| - Z(u), \mathbb{C}) \longrightarrow H^0(|F|, \mathbb{C}) \dashrightarrow \\ \dashrightarrow H^1(|F|, |F| - Z(u), \mathbb{C}) \longrightarrow H^1(|F| - Z(u), \mathbb{C}) \longrightarrow H^1(|F|, \mathbb{C}) \longrightarrow 0. \end{aligned}$$

Note that $H^0(|F|, \mathbb{C}) = \mathbb{C}$ and $H^1(|F| - Z(u), \mathbb{C})$ is always trivial. It follows that

$$H^1_{Z(u)}(|F|, \mathbb{C}),$$

is non-trivial, equal to $\mathbb{C}$, if and only if $|F| = Z(u)$, if and only if $u = (i, j)$, where $i \leq 0$ and $j \leq 0$.

4.5 As in the hint any invertible sheaf $\mathcal{L}$ determines an element $l_{\mathcal{U}}$ of $H^1(\mathcal{U}, \mathcal{O}_X^*)$, where $\mathcal{L}|_{U_i}$ is trivial. If $\mathcal{V}$ is a refinement of $\mathcal{U}$, then $\mathcal{L}|_{V_j}$ is certainly trivial, where $V_j \subset U_i$, and it is easy to check that

$$l_{\mathcal{V}} \in H^1(\mathcal{V}, \mathcal{O}_X^*),$$

is the same as the image of $l_{\mathcal{U}}$ under the natural map

$$H^1(\mathcal{U}, \mathcal{O}_X^*) \longrightarrow H^1(\mathcal{V}, \mathcal{O}_X^*).$$

Thus $\mathcal{L}$ determines an element of the direct limit. Using (5.4) this gives us a map

$$\pi\colon \operatorname{Pic}(X) \longrightarrow H^1(X, \mathcal{O}_X^*).$$

If $\mathcal{L}$ and $\mathcal{M}$ are two invertible sheaves, then there is a common cover $\mathcal{U}$ over which they are both trivial. It is easy to see that the image of $\mathcal{L} \underset{\mathcal{O}_X}{\otimes} \mathcal{M}$ in $H^1(\mathcal{U}, \mathcal{O}_X^*)$ is $l_{\mathcal{U}} + m_{\mathcal{U}}$. But then π is a group homomorphism. To give an element of $H^1(X, \mathcal{O}_X^*)$ is to give an element of $H^1(\mathcal{U}, \mathcal{O}_X^*)$, for some open cover $\mathcal{U}$. Using this 1-cocycle, one can construct an invertible sheaf, $\mathcal{L}$, which represents this 1-cocycle. Thus π is surjective. Suppose that $\mathcal{L}$ is sent to zero. Then there is some open cover $\mathcal{U}$ for which the corresponding 1-cocycle is a coboundary, represented by

$$\sigma_i \in H^0(U_i, \mathcal{O}_X^*).$$

But then σ defines a global non-vanishing section of $\mathcal{L}$, so that

$$\mathcal{L} \simeq \mathcal{O}_X.$$

It follows that π is injective.

4.7 The union of the standard open affine subsets U_1 and U_2 contains every point of $\mathbb{P}^2$ except $[1 : 0 : 0]$. $U_{12} = U_1 \cap U_2$ is affine so that U, V and $U \cap V$ are affine subsets of X. Therefore $\mathcal{U} = \{\, U, V \,\}$ is certainly an open affine cover of X.

Let $x = x_0/x_2$ and $y = x_1/x_2$. Then $u = x_0/x_1 = xy^{-1}$ and $v = x_2/x_1 = y^{-1}$. We want to calculate the cohomology of the complex

$$\frac{k[u,v]}{\langle f(u,1,v)\rangle} \oplus \frac{k[x,y]}{\langle f(x,y,1)\rangle} \longrightarrow \frac{k[x,y,v]}{\langle f(x,y,1)}$$

We already know the kernel consists only of the constants. We check this explicitly.
Suppose that (g,h) is sent to zero. Then

$$g(u,v) - h(x,y) = \alpha f(x,y,1),$$

for some polynomial α in x, y and y^{-1}. Note that the coefficient of x^d is non-zero in f, since $[1:0:0]$ is not a point of the curve. We may write

$$\alpha = \beta + \gamma + \delta,$$

where β is a polynomial whose only non-zero coefficients are of the form $x^i y^j$ where $i \leq -d - j$, for γ we have $j \geq 0$ and δ is what is left over, namely $j < 0$ and $i > -d - j$.
Note that $(g - \beta f, h + \gamma f)$ represents the same element of

$$\frac{k[u,v]}{\langle f(u,1,v)\rangle} \oplus \frac{k[x,y]}{\langle f(x,y,1)\rangle},$$

as (g,h). As βf and γf only have the constant term in common, it is enough to show $\delta = 0$.
Consider the set

$$N = \{\, (i,j) \,|\, \text{the coefficient of } x^i y^j \text{ is non-zero in } \delta \,\}.$$

Pick $(i,j) \in N$ with i maximal and j minimal. Then the coefficient of $x^{i+d}y^j$ is non-zero in $g(u,v) - h(x,y) - \beta f - \gamma f$. So either $i + d \leq -j$ or $j \geq 0$, both of which are impossible as $(i,j) \in N$.
Thus the kernel of the map is the space of constant polynomials.
We consider the cokernel. Each element of the cokernel corresponds to a polynomial in $k[x,y,y^{-1}]$. Monomials of the form $x^i y^j$ where $j \geq 0$ are in the image as are monomials of the form $x^i y^j$, where $i \leq -j$. Since the coefficient of x^d is non-zero in $f(x,y,1)$, it follows that elements of the cokernel are a sum of monomials $x^i y^j$ where $0 \leq i < d$ and $-i < j < 0$. There are

$$\frac{d-1}{d-2}2,$$

such monomials, and we have already checked that none of these monials are in the image of the coboundary map.

7.1 We can split the long exact sequence of cohomology into one short exact sequence,

$$0 \longrightarrow H^0(X, \mathcal{F}') \longrightarrow H^0(X, \mathcal{F}) \longrightarrow Q \longrightarrow 0,$$

and one long exact sequence, which starts with

$$0 \longrightarrow Q' \longrightarrow H^1(X, \mathcal{F}') \longrightarrow H^1(X, \mathcal{F}) \dots,$$

where

$$Q' = \frac{H^0(X, \mathcal{F}'')}{Q}.$$

We have

$$h^0(X, \mathcal{F}) = h^0(X, \mathcal{F}') + \dim_k Q,$$

and, by an obvious induction,

$$\sum_{i \geq 1} (-1)^{i-1} h^i(X, F) = \sum_{i \geq 1} (-1)^{i-1} h^i(X, F') + \sum_{i \geq 0} (-1)^{i-1} h^i(X, F'') - \dim_k Q.$$

Adding the two equations together gives the result.

5.2 (a) Pick a divisor Y belonging to the linear system determined by $\mathcal{O}_X(1)$. Note that there is a morphism of sheaves

$$\mathcal{F}(-1) \longrightarrow \mathcal{F},$$

which is locally given by multiplication by the defining equation of Y, so that this map is an isomorphism away from Y. We get an exact sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{F}(-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{Q} \longrightarrow 0,$$

where $\mathcal{Q}$ and $\mathcal{R}$ are defined to fix exactness. Note that $\mathcal{Q}$ and $\mathcal{R}$ are coherent and they are both supported on Y. If we tensor this exact sequence by $\mathcal{O}_X(n)$ we get

$$0 \longrightarrow \mathcal{R}(n) \longrightarrow \mathcal{F}(n-1) \longrightarrow \mathcal{F}(n) \longrightarrow \mathcal{Q}(n) \longrightarrow 0,$$

By (5.1) we have

$$\Delta\chi(\mathcal{F}(n)) = \chi(\mathcal{Q}(n)) - \chi(\mathcal{R}(n)).$$

By Noetherian induction the RHS is a polynomial and so $\chi(\mathcal{F}(n))$ is also a polynomial.

(b) By Serre vanishing, there is an integer n_0 such that

$$\chi(\mathcal{F}(n)) = h^0(\mathbb{P}^n, \mathcal{F}(n)),$$

for $n \geq n_0$. But we have already seen that the RHS is precisely the dimension of the nth graded piece of $\Gamma_*(\mathcal{F})$.

5.3 (a) If X is integral, and k is an algebraically closed field, then there is a projective variety X' such that $t(X') = X$. We have that

$$H^0(X', \mathcal{O}_{X'}) = H^0(X, \mathcal{O}_X).$$

But by (I.3.4), the LHS is isomorphic to k.
(b) Clear from (5.2).
(c) Let $f\colon C \dashrightarrow X$ be a rational map from a smooth curve to a projective variety. Then f is a morphism. Thus if $f\colon C_1 \dashrightarrow C_2$ is a birational map, then f is in fact an isomorphism. It is then clear that $p_a(C)$ is a birational invariant.
If C is a smooth plane curve of degree d then the arithmetic genus of C is

$$\binom{d-1}{2}.$$

In particular, if $d \geq 3$, the arithmetic genus of C is non-zero, so that C is not rational.
5.7 (a) Let $\mathcal{F}$ be any coherent sheaf on Y. Then $\mathcal{G} = i_*\mathcal{F}$ is a coherent sheaf on X. As $\mathcal{L}$ is ample, there is an integer n_0 such that if $n \geq n_0$, then

$$H^i(X, \mathcal{G} \otimes \mathcal{L}^n) \qquad \text{for any} \qquad n \geq n_0, i > 0.$$

On the other hand,

$$H^i(Y, \mathcal{F} \otimes i^*\mathcal{L}^n) = H^i(X, \mathcal{G} \otimes \mathcal{L}^n).$$

(b) Since X_{red} is a closed subscheme, (a) implies that $\mathcal{L}_{\text{red}}$ is ample. Now suppose that $\mathcal{L}_{\text{red}}$ is ample. Let $\mathcal{F}$ be a quasi-coherent sheaf on X and let $\mathcal{N}$ be the sheaf of nilpotent elements. Then

$$\mathcal{N}^k \cdot \mathcal{L} = 0,$$

for some $k > 0$. Let $\mathcal{G} = \mathcal{N} \cdot \mathcal{F}$. By induction on k, there is a constant n_0 such that

$$H^i(X, \mathcal{G} \otimes \mathcal{L}^n) = 0,$$

for all $n \geq n_0$. There is a short exact sequence,

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow \mathcal{H} \longrightarrow 0,$$

where $\mathcal{H}$ is supported on X_{red}. Possibly increasing n_0, we may assume that

$$H^i(X, \mathcal{H} \otimes \mathcal{L}^n) = 0,$$

for all all $n \geq n_0$. Tensoring by $\mathcal{L}^n$ and taking the long exact sequence of cohomology, we get

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0,$$

all $n \geq n_0$. But then $\mathcal{L}$ is ample by (5.3).
(c) As X_i is a closed subscheme of X, (a) implies that $\mathcal{L} \otimes \mathcal{O}_{X_i}$ is ample. Let $\mathcal{I}$ be the ideal sheaf of X_1. Let $\mathcal{F}$ be a quasi-coherent sheaf. Then there is an exact sequence

$$0 \longrightarrow \mathcal{I} \cdot \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0,$$

where $\mathcal{G}$ is a quasi-coherent sheaf supported on X_1. Tensoring by a sufficiently high power of $\mathcal{L}$ and by induction on the number of irreducible components, taking the long exact sequence of cohomology, we get that

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0,$$

all $n \geq n_0$. But then $\mathcal{L}$ is ample by (5.3).

(d) If $\mathcal{L}$ is ample, and $\mathcal{F}$ is a quasi-coherent sheaf on X, then $f_*\mathcal{F}$ is quasi-coherent sheaf on Y and

$$H^i(X, \mathcal{F} \otimes f^*\mathcal{L}^n) = H^i(Y, f_*\mathcal{F} \otimes \mathcal{L}^n) = 0,$$

for all n sufficiently large. Hence $f^*\mathcal{L}$ is ample.

For the other direction, by (b) and (c) we may suppose that X and Y are integral. Let $\mathcal{F}$ be a quasi-coherent sheaf on Y. As in the proof of (4.2), we may find an exact sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow f_*\mathcal{G} \longrightarrow \mathcal{F}^r \longrightarrow \mathcal{Q} \longrightarrow 0,$$

where $\mathcal{R}$ and $\mathcal{Q}$ are quasi-coherent sheaves, supported on proper closed subsets of Y, and $\mathcal{G}$ is a coherent sheaf on X. Tensoring by a high power of $\mathcal{L}$, applying Noetherian induction, we get

$$H^i(Y, \mathcal{F}^r \otimes \mathcal{L}^n) = H^i(X, \mathcal{G} \otimes f^*\mathcal{L}^n) = 0,$$

for all $i > 0$. But then $\mathcal{L}$ is ample.