MODEL ANSWERS TO HWK #4

1. (i) It suffices to check that the determinant of the primitive generators of every maximal cone is one or minus one. The maximal cones are given by

$$\begin{array}{cccc}
\langle v_1, v_2, v_3\rangle & \langle v_1, v_2, v_4\rangle & \langle v_2, v_4, v_5\rangle & \langle v_2, v_3, v_5\rangle \\
\langle v_3, v_5, v_6\rangle & \langle v_1, v_3, v_6\rangle & \langle v_1, v_4, v_6\rangle & \langle v_4, v_5, v_7\rangle \\
\langle v_4, v_6, v_7\rangle & \langle v_5, v_6, v_8\rangle & \langle v_5, v_7, v_8\rangle & \langle v_6, v_7, v_8\rangle,
\end{array}$$

and probably the most sensible way to compute the determinants is to use a computer algebra system.

(ii) As $|D|$ is base point free, it contains a T-Cartier divisor $D' \geq 0$. Replacing D by D', we may assume that $D \geq 0$ and we will show that then $D = 0$. Suppose that $D = \sum d_i D_i$.

Let $\phi = \phi_D$ be the continuous, piecewise linear integral function associated to D. As $v_5 = v_2 + v_4 - v_1$, $\langle v_1, v_2, v_4\rangle$ is a cone and ϕ is convex

$$d_1 + d_5 \geq d_2 + d_4.$$

As $v_2 + v_6 = 2(v_3 + v_5)$, $\langle v_2, v_3, v_5\rangle$ is a cone and ϕ is convex,

$$d_2 + d_6 \geq 2(d_3 + d_5).$$

Finally, as $v_3 + v_4 = 2v_1 + v_6$, $\langle v_1, v_3, v_6\rangle$ is a cone and ϕ is convex,

$$d_3 + d_4 \geq 2d_1 + d_6.$$

Adding these inequalities together we get

$$d_1 + d_2 + d_3 + d_4 + d_5 + d_6 \geq 2d_1 + d_2 + 2d_3 + d_4 + 2d_5 + d_6.$$

As $d_i \geq 0$, it follows that $d_1 = d_3 = d_5 = 0$. By the first inequality, $d_2 = d_4 = 0$ (and by the last inequality $d_6 = 0$). As every vector in $\mathbb{R}^3$ is a positive linear combination of v_1, v_2, v_3, v_4 and v_5, it follows that the polytope P_D associated to D is the zero polytope. As D is base point free this is only possible if $D = 0$.

(iii) Clear, since if X is projective then there is a very ample divisor on X which is not linearly equivalent to zero.

7.8 A section $\sigma\colon X \longrightarrow \mathbb{P}(\mathcal{E})$ is the same as a morphism of X to $\mathbb{P}(\mathcal{E})$ over X. But we already know that this is the same as the data of an invertible sheaf $\mathcal{L}$ and a surjective morphism $\mathcal{E} \longrightarrow \mathcal{L}$.

7.9 (a) As stated this result is trivially false. Take X be the disjoint union of two points, then P is the disjoint union of two copies of $\mathbb{P}^n$

and the Picard group of P is $\mathbb{Z} \oplus \mathbb{Z}$, not $\mathbb{Z}$. So we assume that X is connected.

Let $P = \mathbb{P}(\mathcal{E})$. It is sufficient to prove that

$$\operatorname{Pic}(P) = \pi^* \operatorname{Pic}(X) \oplus \mathbb{Z}\langle \mathcal{O}_P(1) \rangle.$$

Note that if $\mathcal{L}$ is any invertible sheaf on X then $\pi^*\mathcal{L}$ restricts to the trivial line bundle on any fibre. Since $\operatorname{Pic}(\mathbb{P}^n) = \mathbb{Z}$ is generated by $\mathcal{O}_{\mathbb{P}^n}(1)$, it follows that if $\pi^*\mathcal{L}(k) \simeq \mathcal{O}_P$, then $k = 0$. But

$$\pi_*\pi^*\mathcal{L} = \mathcal{L} \otimes \pi_*\mathcal{O}_P = \mathcal{L},$$

by push-pull, so that $\mathcal{L} \simeq \mathcal{O}_P$. Thus the RHS is a subgroup of the LHS.

To finish off, we need to prove that if we have an invertible sheaf $\mathcal{M}$ on P which restricts to the trivial sheaf on one fibre then it is the pullback of a sheaf from X. Now if $P = X \times \mathbb{P}^n$ and X is regular and separated, then

$$\operatorname{Cl}(P) \simeq \operatorname{Cl}(X) \times \mathbb{Z}.$$

As X is regular and X is separated, Cartier divisors are the same as Weil divisors, and so

$$\operatorname{Pic}(P) = \pi^* \operatorname{Pic}(X) \times \mathbb{Z}.$$

Hence if $\mathcal{M}$ is trivial over the point p then there is an open neighbourhood of p such that $\mathcal{M}$ restricts to the trivial line bundle on every fibre. As X is connected, it follows that $\mathcal{M}$ is trivial on every fibre. By what we just observed this implies that $\mathcal{M}$ is locally the pullback of a line bundle. Let $\mathcal{L} = \pi_*\mathcal{M}$. Then $\mathcal{L}$ is a line bundle, since we can check this locally. Consider the induced morphism of line bundles

$$\pi^*\mathcal{L} \longrightarrow \mathcal{M}.$$

This morphism is surjective, since it is surjective locally and so it is an isomorphism.

(b) One direction is easy. If $\mathcal{E}' = \mathcal{E} \otimes \mathcal{L}$ then $\mathcal{S}' = \mathcal{S} \star \mathcal{L}$ and we have already seen that P and P' are isomorphic over X.

Now suppose that P and P' are isomorphic over X. As

$$\mathcal{O}_{P'}(1) \in \operatorname{Pic}(P') \simeq \operatorname{Pic}(P),$$

by what we have already proved

$$\mathcal{O}_{P'}(1) \simeq \pi^*\mathcal{L} \otimes \mathcal{O}_P(k),$$

for some line bundle on X and some integer k. Restricting to a fibre, it follows easily that $k = 1$. If we push this equation down to X we get

$$\mathcal{E}' \simeq \mathcal{E} \otimes \mathcal{L},$$

by push-pull.
7.10 (a) A **projective n-space bundle** over X is a morphism of schemes $\pi\colon P \longrightarrow X$ together with an open cover $\{U_i\}$ and isomorphisms $\psi_i\colon \pi^{-1}(U_i) \longrightarrow \mathbb{P}^n_{U_i}$ such that for every open affine $V = \operatorname{Spec} A \subset U_i \cap U_j$ the automorphism $\psi = \psi_j \circ \psi_i^{-1}\colon \mathbb{P}^n_V \longrightarrow \mathbb{P}^n_V$ is given by a **linear** automorphism θ of $A[x_0, x_1, \dots, x_n]$, that is, $\theta(a) = a$ for every $a \in A$ and $\theta(x_i) = \sum a_{ij} x_j$ for suitable constants (a_{ij}).
A **isomorphism** of two projective space bundles $(P, \pi, \{U_i\}, \{\psi_i\})$ and $(P', \pi', \{U'_i\}, \{\psi'_i\})$ is an isomorphism of P to P' over X, such that over any affine subset $V \subset U_i \cap U'_j$ the induced automorphism $\psi = \psi_i \cap \psi_j'^{-1}$ is given by a linear automorphism θ of $A[x_0, x_1, \dots, x_n]$.
(b) By assumption there is an open cover $\{U_i\}$ such that $\mathcal{E}|_{U_i}$ is free of rank $n+1$. In this case $\mathbb{P}(\mathcal{E}|_{U_i}) = \mathbb{P}^n_{U_i}$. By assumption, if $V \subset U_i \cap U_j$ then the induced linear map of affine bundles is linear.
(c) As in the hint, pick an open subset U over which P is isomorphic to $\mathbb{P}^n_U$ and let $\mathcal{L}_0$ be $\mathcal{O}_{\mathbb{P}^n_U}(1)$. Let $H_0 \subset \mathbb{P}^n_U$ be a hyperplane and let H be its closure in P. Then H has codimension one in P. As X is locally separated, and X is regular, in fact H defines a Cartier divisor. Let $\mathcal{L} = \mathcal{O}_P(H)$ be the associated invertible sheaf. Clearly $\mathcal{L}|_{\pi^{-1}(U)} = \mathcal{L}_0$. Arguing as in (7.9) (a) it follows that $\mathcal{L}$ restricts to $\mathcal{O}(1)$ on every fibre. Let $\mathcal{E} = \pi_*\mathcal{L}$. Then $\mathcal{E}$ is locally free of rank $n+1$. Indeed this can be checked locally, in which case P is a product and the result is clear. Let $P' = \mathbb{P}(\mathcal{E})$. Now there is a morphism

$$\pi^*\mathcal{E} \longrightarrow \mathcal{L},$$

which is surjective, as this can be checked locally. But then there is a morphism $P \longrightarrow P'$ over X. But then this map is an isomorphism, as it is an isomorphism locally over X.
(d) Easy consequence of (7.9) (b), (b) and (c).
7.11 (a) By the universal property, it suffices to check this locally. So we may assume that $X = \operatorname{Spec} A$ is an affine scheme. Let $I = H^0(X, \mathcal{I})$. Then $Y = \operatorname{Proj} S$, where

$$S = \bigoplus_{m=0}^{\infty} I^m.$$

and $Y' = \operatorname{Proj} S_{(d)}$. But we have already seen that Y and Y' are then isomorphic over X.
(b) One way to prove this is to observe that

$$\mathcal{S}' = \mathcal{S} \star \mathcal{J}.$$

Another is to observe that if $g\colon Z \longrightarrow X$ is any morphism then

$$g^{-1}(\mathcal{I} \cdot \mathcal{J}) \cdot \mathcal{O}_Z = (g^{-1}\mathcal{I} \cdot \mathcal{O}_Z) \cdot (g^{-1}\mathcal{J} \cdot \mathcal{O}_Z).$$

Since

$$g^{-1}\mathcal{J} \cdot \mathcal{O}_Z,$$

is always an invertible sheaf, it follows that

$$g^{-1}(\mathcal{I} \cdot \mathcal{J}) \cdot \mathcal{O}_Z,$$

is an invertible sheaf if and only if

$$g^{-1}\mathcal{I} \cdot \mathcal{O}_Z,$$

is an invertible sheaf. But then the blow up of $\mathcal{I}$ and the blow up of $\mathcal{I} \cdot \mathcal{J}$ satisfy the same universal property, so that they are isomorphic.
(c) Pick a very ample divisor H on Z, whose support does not contain any fibre of f. Let $D = \pi(H)$. Then a priori D determines a Weil divisor but as X is regular it is a Cartier divisor. Then H is equal to the strict transform of D, so that $E = \pi^*D - H \geq 0$ and E is exceptional for f (that is, its image has codimension at least two).
By assumption $-E$ is relatively very ample. Let $\mathcal{I} = f_*\mathcal{O}_Z(-E)$. Then $\mathcal{I} \subset \mathcal{O}_X$ is a coherent $\mathcal{O}_X$-module, that is, a coherent ideal sheaf. As E is relatively very ample, the morphism of sheaves

$$f^*f_*\mathcal{O}_X(-E) \longrightarrow \mathcal{O}_Z(-E),$$

is surjective. It follows that

$$f^{-1}\mathcal{I} \cdot \mathcal{O}_Z \longrightarrow \mathcal{O}_Z(-E),$$

is surjective. As $f^{-1}\mathcal{I} \cdot \mathcal{O}_Z$ is a coherent ideal sheaf, it follows that $f^{-1}\mathcal{I} \cdot \mathcal{O}_Z = \mathcal{O}_Z(-E)$. In particular $f^{-1}\mathcal{I} \cdot \mathcal{O}_Z$ is an invertible sheaf. As

$$Z = \mathbf{Proj} \bigoplus_{m=0}^{\infty} \pi_*\mathcal{O}_Z(-mE) = \mathbf{Proj} \bigoplus_{m=0}^{\infty} \mathcal{I}^m,$$

it follows that Z is the blow up of $\mathcal{I}$. Let V be the image of E. Then the subscheme of X defined by $\mathcal{I}$ is supported on V. On the other hand, V is contained in $X - U$ as E is a divisor and V is not.
7.12. Presumably this question should be slightly reworded to say that no irreducible component of Y is contained in an irreducible component of Z and vice-versa.
This problem is local (see above), so we might as well assume that $X = \operatorname{Spec} A$ is affine. In this case Y and Z are defined by ideals I and J. Let $K = I + J$ the ideal of the intersection. Then

$$Y = \operatorname{Proj} S = \bigoplus_{d=0}^{\infty} K^d,$$

is the blow up of $Y \cap Z$. We just need to check that the strict transforms $\tilde{Y}$ and $\tilde{Z}$ of Y and Z don't intersect on the exceptional divisor of the blow up. Pick generators $a_1, a_2, \dots, a_n$ for the ideal K. We may suppose that $a_1, a_2, \dots, a_m$ are generators of the ideal I and that the rest generate the ideal J. This defines a surjective ring homomorphism

$$\phi\colon A[x_1, x_2, \dots, x_n] \longrightarrow S,$$

of graded rings, just by sending x_i to a_i. The defines a closed embedding $Y \subset \mathbb{P}^n_A$. Note that the kernel of ϕ contains the polynomials $a_j x_i - a_i x_i$. Suppose we are given a point p of $Y - Y \cap Z$. Then we may find $j > m$ such that a_i does not vanish at p. If $i \leq m$ then x_i must vanish in the fibre over p since a_i vanishes but a_j does not. Therefore $x_1, x_2, \dots, x_m$ vanish on $\tilde{Y}$, since this is the closure of the inverse image of $Y - Y \cap Z$ and by symmetry the rest of the variables vanish on $\tilde{Z}$. But then $\tilde{Y}$ and $\tilde{Z}$ don't intersect.
7.13. (a) Let U_0 and U_1 be the two standard open affine subsets of $\mathbb{P}^1$. Define two morphisms,

$$C \times U_0 \longrightarrow C \times U_0 \qquad \text{and} \qquad C \times U_0 - \{[1:0]\} \longrightarrow C \times U_0,$$

where the first morphism is the identity and the second morphism is given by $(P, u) \longrightarrow (\phi_u(P), u)$. These two morphisms glue to a morphism $\pi^{-1}(U_0) \longrightarrow C \times U_0$, which is easily seen to be an isomorphism. Hence $\pi^{-1}(U_i) \simeq C \times U_i$ and π is nothing more than projection onto the second factor. As properness is local on the base, π is certainly proper. As the composition of proper morphisms is proper, X is complete.
(b) Let $\pi\colon \tilde{Y} \longrightarrow Y$ be the normalisation of a variety Y. As π is birational $\pi_*\mathcal{K}_{\tilde{Y}} = \mathcal{K}$. Thus there is a natural surjective morphism of sheaves

$$\mathcal{K}^* \longrightarrow \mathcal{K}^*/\pi_*\mathcal{O}^*_{\tilde{Y}}.$$

As

$$\mathcal{O}_Y \subset \pi_*\mathcal{O}_{\tilde{Y}},$$

this induces a surjective morphism

$$\mathcal{K}^*/\mathcal{O}^*_Y \longrightarrow \mathcal{K}^*/\pi_*\mathcal{O}^*_{\tilde{Y}}.$$

Hence there is a sequence

$$0 \longrightarrow \pi_*\mathcal{O}^*_{\tilde{Y}}/\mathcal{O}^*_Y \longrightarrow \mathcal{K}^*/\mathcal{O}^*_Y \longrightarrow \mathcal{K}^*/\pi_*\mathcal{O}^*_{\tilde{Y}} \longrightarrow 0,$$

which is clearly exact, as it is exact on stalks. If we take global sections, then we get an exact sequence

$$0 \longrightarrow H^0(Y, \pi_*\mathcal{O}^*_{\tilde{Y}}/\mathcal{O}^*_Y) \longrightarrow H^0(Y, \mathcal{K}^*/\mathcal{O}^*_Y) \longrightarrow H^0(Y, \mathcal{K}^*/\pi_*\mathcal{O}^*_{\tilde{Y}}).$$

For the third term we have

$$H^0(Y, \mathcal{K}^*/\pi_*\mathcal{O}^*_{\tilde{Y}}) = H^0(\tilde{Y}, \mathcal{K}^*/\mathcal{O}^*_{\tilde{Y}}).$$

So the second and third terms are nothing but the group of Cartier divisors on Y and $\tilde{Y}$. If we mod out by linear equivalence, that is, by the group

$$H^0(Y, \mathcal{K}^*),$$

then the second and third terms become the Picard groups of Y and $\tilde{Y}$. So there is an exact sequence

$$0 \longrightarrow H^0(Y, \pi_* \mathcal{O}^*_{\tilde{Y}}/\mathcal{O}^*_Y) \longrightarrow \operatorname{Pic}(Y) \longrightarrow \operatorname{Pic}(\tilde{Y}).$$

We apply this in two situations, to $Y = C \times \mathbb{A}^1$ and $Y = C \times (\mathbb{A}^1 - \{0\})$. In both cases $\operatorname{Pic}(\tilde{Y}) = \mathbb{Z}$, since in the first case $\tilde{Y} = \mathbb{P}^1 \times \mathbb{A}^1$ and in the second case $\tilde{Y} = \mathbb{P}^1 \times (\mathbb{A}^1 - \{0\})$. Consider

$$H^0(Y, \pi_* \mathcal{O}^*_{\tilde{Y}}/\mathcal{O}^*_Y).$$

The sheaf

$$\pi_* \mathcal{O}^*_{\tilde{Y}}/\mathcal{O}^*_Y,$$

is supported on $p \times \mathbb{A}^1$, or $p \times (\mathbb{A}^1 - \{0\})$, as appropriate, where p is the node. As a sheaf on $\mathbb{A}^1$ it is isomorphic to $\mathcal{O}^*_{\mathbb{A}^1}$. As observed in the hint,

$$H^0(\mathbb{A}^1, \mathcal{O}^*_{\mathbb{A}^1}) = \mathbb{G}_m \qquad \text{and} \qquad H^0(\mathbb{A}^1 - \{0\}, \mathcal{O}^*_{\mathbb{A}^1}) = \mathbb{G}_m \times \mathbb{Z}.$$

Thus

$$\operatorname{Pic}(C \times \mathbb{A}^1) = \mathbb{G}_m \times \mathbb{Z} \qquad \text{and} \qquad \operatorname{Pic}(C \times (\mathbb{A}^1 - \{0\})) = \mathbb{G}_m \times \mathbb{Z}^2.$$

(c) Projection $C \times \mathbb{A}^1 \longrightarrow C$ to the first factor defines a map on invertible sheaves by pullback, which induces an isomorphism

$$\operatorname{Pic}(C) \simeq \operatorname{Pic}(C \times \mathbb{A}^1).$$

Similarly pullback defines an injective map

$$\operatorname{Pic}(C) \longrightarrow \operatorname{Pic}(C \times (\mathbb{A}^1 - \{0\})),$$

which sends $\langle t, n \rangle$ to $\langle t, 0, n \rangle$. Thus the natural restriction map

$$\operatorname{Pic}(C \times \mathbb{A}^1) \longrightarrow \operatorname{Pic}(C \times (\mathbb{A}^1 - \{0\})),$$

has the same form. Now let us consider the action of ϕ, on $\operatorname{Pic}(Y)$,

$$\phi^* \colon \operatorname{Pic}(Y) \longrightarrow \operatorname{Pic}(Y).$$

It suffices to determine

$$\phi^*(t, 0, 0), \qquad \phi^*(0, 1, 0) \qquad \text{and} \qquad \phi(0, 0, 1).$$

As $\mathbb{G}_m$ is a connected algebraic group and $\mathbb{Z}$ is a discrete group, every group homomorphism

$$\mathbb{G}_m \longrightarrow \mathbb{Z},$$

is trivial. On the other hand, multiplication by $a \in \mathbb{G}_m$ induces the identity on $\mathrm{Pic}(C)$. It is not hard to see from this that

$$\phi^*(t, 0, 0) = \langle t, 0, 0 \rangle.$$

Now the isomorphism

$$H^0(Y, \pi_* \mathcal{O}^*_{\tilde{Y}} / \mathcal{O}^*_Y) \simeq H^0(\mathbb{A}^1 - \{0\}, \mathcal{O}^*_{\mathbb{A}^1}),$$

sends $f \in \mathcal{O}^*_{\tilde{Y}}$ to the ratio of f at the two points $p_0 = [1 : 0]$ and $p_1 = [0 : 1]$ lying over p. The line bundle $\langle 0, 1, 0 \rangle$ corresponds to f which takes on the value u at p_0 and 1 at p_1. The action of ϕ fixes f and from this it is clear that

$$\phi^*(0, 1, 0) = \langle 0, 1, 0 \rangle.$$

Finally consider the line bundle corresponding to $\langle 0, 0, 1 \rangle$. This corresponds to the line bundle $\mathcal{O}_{\mathbb{P}^1}(1)$ on $\mathbb{P}^1$, pulled back to $\tilde{Y} = \mathbb{P}^1 \times (\mathbb{A}^1 - \{0\})$. The corresponding line bundle is given by x on $U_0 \times (\mathbb{A}^1 - \{0\})$ and 1 on $U_1 \times (\mathbb{A}^1 - \{0\})$. Applying ϕ we get ux on $U_0 \times (\mathbb{A}^1 - \{0\})$ and 1 on $U_1 \times (\mathbb{A}^1 - \{0\})$. The line bundle with these transition functions is $\langle 0, 1, 1 \rangle$. Putting all of this together, we see that

$$\phi^*(t, d, n) = \langle t, d + n, n \rangle.$$

(d) Let $\mathcal{L}$ be an invertible sheaf on X. If we restrict $\mathcal{L}$ to $C \times U_0$ then we get an element $\langle t, n \rangle$ of $\mathrm{Pic}(C \times U_0)$ and if we restrict to $C \times U_1$ then we get another element $\langle s, m \rangle$ of $\mathrm{Pic}(C \times U_1)$. Their images in $\mathrm{Pic}(C \times (U_0 \cap U_1))$ are $\langle t, 0, n \rangle$ and $\langle s, m, m \rangle$. Since these are supposed to agree, we must have $s = t$ and $m = n = 0$. But then the restriction of $\mathcal{L}$ to $C \times \{0\}$ has degree zero, so $\mathcal{L}$ cannot be ample. In particular X is not projective over k and π is not projective.