MODEL ANSWERS TO HWK #3

1. Let K be the field of fractions of A. Then
$$K = \frac{k(x_1, x_2, \dots, x_n)[z]}{\langle z^2 - f\rangle}.$$
This is a quadratic extension of the field $L = k(x_1, x_2, \dots, x_n)$. As the characteristic is not 2, K is the splitting field of $z^2 - f$ so that K/L is Galois, with Galois group $\mathbb{Z}/2\mathbb{Z}$ given by the involution $z \longrightarrow -z$.
Every element α of K is uniquely of the form $g + hz$, where g and $h \in k(x_1, x_2, \dots, x_n)$. Then the conjugate β of α is $g - hz$ so that
$$(X - \alpha)(X - \beta) = X^2 - (\alpha + \beta)X + (\alpha\beta) = X^2 - 2gX + (g^2 - h^2 f),$$
is the minimal polynomial of α. α is in the integral closure of $k[x_1, x_2, \dots, x_n]$ inside K if and only if $2g$ and $g^2 - h^2 f \in k[x_1, x_2, \dots, x_n]$. But $2g \in k[x_1, x_2, \dots, x_n]$ if and only if $g \in k[x_1, x_2, \dots, x_n]$. In this case $g^2 - h^2 f \in k[x_1, x_2, \dots, x_n]$ if and only if $h^2 f \in k[x_1, x_2, \dots, x_n]$. As f is square free and $k[x_1, x_2, \dots, x_n]$ is a UFD this happens if and only if $h \in k[x_1, x_2, \dots, x_n]$. But then A is the integral closure of $k[x_1, x_2, \dots, x_n]$.
In particular A is integrally closed.
2. (a) Note that if $r \geq 2$ then $x_0^2 + x_1^2 + x_2^2 + \dots + x_r^2$ is irreducible, as the characteristic is not two. In particular it is square free and we may apply (II.6.4).
(b) As k is algebraically closed there is an element i such that $i^2 + 1 = 0$. Consider the change of variables which replaces x_0 by ix_0 and fixes the other variables. This has the effect of replacing
$$x_0^2 + x_1^2 + x_2^2 + \dots + x_r^2 \qquad \text{by} \qquad -x_0^2 + x_1^2 + x_2^2 + \dots + x_r^2.$$
Now consider the change of variables which sends
$$2x_0 \longrightarrow x_0 + x_1 \qquad \text{and} \qquad 2x_1 \longrightarrow x_0 - x_1,$$
and fixes the other variables. As
$$x_1^2 - x_0^2 = (x_0 + x_1)(x_1 - x_0),$$
this has the effect of replacing
$$-x_0^2 + x_1^2 + x_2^2 + \dots + x_r^2 \qquad \text{by} \qquad x_0 x_1 + x_2^2 + \dots + x_r^2.$$
Finally multiplying x_0 by -1 we can put the equation for X into the form
$$x_0 x_1 = x_2^2 + \dots + x_r^2.$$

(1) We have $x_0x_1 = x_2^2$. This is essentially done in example (II.6.5.2) and we closely follow the treatment there. Let L be the linear space $x_1 = x_2 = 0$ in $\mathbb{A}^{n+1}_k$ and let $U = X - L$. As L is a prime divisor in X, we have

$$\mathbb{Z} \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U) \longrightarrow 0.$$

Now $2L$ is Cartier, defined by $x_1 = 0$; if $x_1 = 0$ then $x_2^2 = 0$ and x_2 has multiplicity one along L. It follows that

$$\begin{aligned} U &= \operatorname{Spec} \frac{k[x_0, x_1, x_2, \dots, x_n]_{x_1}}{\langle x_0x_1 - x_2^2\rangle} \\ &= \operatorname{Spec} k[x_1, x_1^{-1}, x_2, x_3, \dots, x_n] \\ &\simeq \mathbb{G}_m \times \mathbb{A}^{n-1}_k, \end{aligned}$$

since we may write $x_0 = x_1^{-1}x_2^2$ in the ring $k[x_0, x_1, x_2, \dots, x_n]_{x_1}$. Hence $\mathrm{Cl}(U) = 0$.

It remains to show that L is not Cartier. This is equivalent to showing that the ideal $\mathfrak{p} = \langle x_1, x_2\rangle$ is not principal. Let $\mathfrak{m} = \langle x_0, x_1, x_2\rangle$ be the maximal ideal of the vertex of the cone. Then $\mathfrak{p} \subset \mathfrak{m}$ and the image of $\mathfrak{p}$ inside the quotient

$$\frac{\mathfrak{m}}{\mathfrak{m}^2},$$

is two dimensional, as the images x_1 and x_2 are independent. Thus $\mathfrak{p}$ is not principal and Y is not Cartier.

(2) We have $x_0x_1 = x_2^2 + x_3^2$. Then $X = Y \times \mathbb{A}^{n-4}_k$ where $Y \subset \mathbb{A}^4_k$ has the same equation. So we may assume that $n = 3$.

Let $V \subset \mathbb{P}^3_k$ be the quadric with equation $X_0X_1 = X_2^2 + X_3^2$. Then $V \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and by (II.6.6.1) $\mathrm{Cl}(V) = \mathbb{Z} \oplus \mathbb{Z}$. By (II.6.3.b) there is an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Cl}(V) \longrightarrow \mathrm{Cl}(X) \longrightarrow 0,$$

where the first map sends 1 to a hyperplane section of $\mathrm{Cl}(V)$. The image of 1 is therefore $(1, 1) \in \mathbb{Z} \oplus \mathbb{Z}$ and the quotient is $\mathbb{Z}$.

(3) Note that the hyperplane $x_1 = 0$ intersects X in the closed set Z defined by $x_2^2 + x_3^2 + \cdots + x_r^2$, which is irreducible. Let U be the complement. Consider projection down to $\mathbb{P}^{n-1}_k$, from the point $[1 : 0 : 0 : \cdots : 0]$. Let $V \simeq \mathbb{A}^{n-1}_k \subset \mathbb{P}^{n-1}_k$ be the standard open subset where $X_1 \neq 0$. Given $[a_1 : a_2 : \cdots : a_n] \in V$, note that there is a unique point

$$a_0 = \frac{-1}{a_1}(a_2^2 + a_3^2 + \dots a_n^2),$$

such that $[a_0 : a_1 : \cdots : a_n] \in U$ projects down to V. It follows easily that $V \simeq U = \mathbb{A}^{n-1}_k$. In particular $\mathrm{Cl}(U) = 0$. On the other hand

Z is linearly equivalent to zero so that $\mathrm{Cl}(X) = 0$ by the usual exact sequence.
(c) Using (II.6.3.a) $n - r$ times, we reduce to the case when $r = n$.
(1) In this case, we have a smooth conic in $\mathbb{P}^2_k$. Any such is isomorphic to $\mathbb{P}^1_k$ (stereographic projection) and so $\mathrm{Pic}(Q) = \mathbb{Z}$. It is clear that a line in $\mathbb{P}^2_k$ intersects the conic in two points.
(2) This is (II.6.6.1).
(3) By (II.6.3.b) there is an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Cl}(Q) \longrightarrow \mathrm{Cl}(X) \longrightarrow 0.$$

The last group is zero, so that $\mathrm{Cl}(Q) \simeq \mathbb{Z}$.
(d) We already know that the homogeneous coordinate ring of Q is integrally closed and that the class group of the corresponding affine variety is zero. It follows that the homogeneous coordinate ring of Q is a UFD by (II.6.2).
Let I be the ideal of Y in $\mathbb{P}^n_k$. Then I contains a homogeneous polynomial F which is not a multiple of $X_0^2 + X_1^2 + \cdots + X_r^2$. Let f be the image of F inside the homogeneous coordinate ring $S(Q)$. We may factor f as $f_1^{a_1} \dots f_m^{a_m}$, where $f_1, f_2, \dots, f_m$ are homogeneous and irreducible. As Y is irreducible, one of the factors, say f_1 vanishes on Y. As f_1 is irreducible, its zero locus is irreducible and reduced, so that the zero locus of f_1 is Y, as a scheme.
Suppose that F_1 is a homogeneous polynomial which represents f_1. Let V be the hypersurface defined by F_1. Then $V \cap Q = Y$, by choice of f_1.
3. (b) We may assume that $r = n$.
(1) Let $\sigma \subset \mathbb{R}^2$ be the cone spanned by $2e_1 - e_2$ and e_2. Then

$$U_\sigma = \mathrm{Spec}\, \frac{K[u, v, w]}{\langle v^2 - uw \rangle}.$$

We have already seen that the class group is $\mathbb{Z}_2$ in class.
(2) Note that $x_0x_1 = x_2^2 + x_3^2$ is equivalent to $x_0x_1 = x_2x_3$. If $r = 3$ we have the cone spanned by four vectors v_1, v_2, v_3 and v_4 such that $v_1 + v_3 = v_2 + v_4$ and any three vectors span the lattice N. Let D_1, D_2, D_3 and D_4 be the four invariant divisors corresponding to v_1, v_2, v_3 and v_4. We may take $v_1 = e_1$, $v_2 = e_2$, $v_3 = e_3$, so that $v_4 = e_1 + e_3 - e_2$. We have an exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^4 \longrightarrow \mathrm{Cl}(X) \longrightarrow 0,$$

where $M = \mathbb{Z}^3$. Let $x = \chi^{(1,0,0)}$, $y = \chi^{(0,1,0)}$ and $z = \chi^{(0,0,1)}$. Then

$$(x) = D_1 + D_4 \qquad (y) = D_2 - D_4 \qquad \text{and} \qquad (z) = D_3 + D_4.$$

Thus D_1, D_2 and D_3 are all multiples of D_4, so that the class group is $\mathbb{Z}$.
(c) Arguing as before, we may assume that $r = n$. The case $r = 2$ is easy, we have $\mathbb{P}^1$.
If $r = 3$, then let F be the cone with support $\mathbb{R}^2$, given by the three vectors $v_1 = e_1$, $v_2 = e_2$ and $v_3 = -2e_1 - e_2$. We have already seen that in a previous hwk that this gives the quadric cone in $\mathbb{P}^3$. As usual there are three invariant divisors D_1, D_2 and D_3, corresponding to the three vectors v_1, v_2 and v_3. Computing as usual, there are two relations

$$D_1 - 2D_3 = 0 \qquad \text{and} \qquad D_2 - D_3 = 0.$$

So every divisor is a multiple of D_3 and the class group is $\mathbb{Z}$.
4. There are many ways to prove this result. Just for practice, let's use the language of continuous, piecewise integral linear functions. We will also make some standard reductions, some of which are superfluous.
As every Weil divisor is linearly equivalent to an invariant Weil divisor, it suffices to check that every invariant Weil divisor is $\mathbb{Q}$-Cartier if and only if every cone is simplicial.
As this result is local, we may assume that $X = U_\sigma$ is an affine toric variety. In this case it is enough to check when σ is simplicial, since every face of a simplicial cone is simplicial. If σ does not span $N_\mathbb{R}$ then $U_\sigma = U_\tau \times \mathbb{G}_m^k$, for some affine toric variety U_τ. In this case the class group of U_σ is equal to the class group of U_τ. So we may assume that σ spans $N_\mathbb{R}$. Let $v_1, v_2, \dots, v_m$ be the primitive generators of the one dimensional faces of σ. Then $m \geq n$ and σ is simplicial if and only if $m = n$.
As $X = U_\sigma$ is an affine toric variety, T-Cartier divisors correspond to integral linear functions $\phi\colon \sigma \longrightarrow \mathbb{R}$ (note that we can drop the word piecewise, so that we can drop the word continuous).
Suppose that $m = n$. Given an invariant Weil divisor $D = \sum a_i D_i$, consider the vectors $w_1, w_2, \dots, w_n$, where $w_i = v_i/a_i$. These belong to an affine hyperplane $H \subset N_\mathbb{R} = \mathbb{R}^n$, which does not contain the origin. Let $\phi\colon \sigma \longrightarrow \mathbb{R}$ be the linear function which takes the value 1 on H. Then $\phi(v_i) = a_i$ and there is a postive integer m such that $m\phi$ is integral. Thus D is T-Cartier and so X is $\mathbb{Q}$-factorial.
Now suppose that X is $\mathbb{Q}$-factorial. Suppose that $m > n$. Then $v_1, v_2, \dots, v_m$ are linearly dependent. Suppose that

$$\sum \lambda_i v_i = 0.$$

We may suppose that $\lambda_1, \lambda_2, \dots, \lambda_m$ are rational. Clearing denominators, we may assume that $\lambda_1, \lambda_2, \dots, \lambda_m$ are integers. Possibly relabelling, we may assume that

$$a_m v_m = \sum a_i v_i,$$

where $a_m \neq 0$ and $a_1, a_2, \dots, a_{m-1}$ are integers. Let $D = \sum_{i \leq m-1} a_i D_i$. If ϕ is a linear function such that $\phi(v_i) = -a_i$ then $\phi(v_m) = -a_m$. So no multiple of D is Cartier.

5. (a) We just have to show that if $\mathcal{L}$ and $\mathcal{M}$ are two invertible sheaves then

$$f^*(\mathcal{L} \underset{\mathcal{O}_X}{\otimes} \mathcal{M}) \simeq f^*\mathcal{L} \underset{\mathcal{O}_Y}{\otimes} f^*\mathcal{M}$$

We exhibit an isomorphism of $\mathcal{O}_Y$-modules

$$f^*(\mathcal{L} \underset{\mathcal{O}_X}{\otimes} \mathcal{M}) \longrightarrow f^*\mathcal{L} \underset{\mathcal{O}_Y}{\otimes} f^*\mathcal{M}.$$

By the adjoint property of f_* and f^* (see page 110 of Hartshorne), it suffices to exhibit an isomorphism of $\mathcal{O}_X$-modules

$$\mathcal{L} \underset{\mathcal{O}_X}{\otimes} \mathcal{M} \longrightarrow f_*(f^*\mathcal{L} \underset{\mathcal{O}_Y}{\otimes} f^*\mathcal{M}).$$

Such an isomorphism is given by the projection formula, see (II.5.1.iv).

(b) Since both maps are linear, it suffices to check that if $q \in Y$ is a closed point then

$$f^*\mathcal{O}_Y(q) \simeq \mathcal{O}_X(f^*q).$$

Both sides live naturally inside $\mathcal{K}$, the sheaf of total quotient rings on X. In this case case we just want to check that

$$f^*\mathcal{O}_Y(q) = \mathcal{O}_X(f^*q),$$

as subsheaves of $\mathcal{K}$.

We check this stalk by stalk. If $p \in X$ is a point not in the fibre of q then both sides are the stalk of the sheaf of regular functions $\mathcal{O}_{X,p}$, so that equality is clear. Now suppose that $f(p) = q$. Note that $\mathcal{O}_{Y,q}(q)$ is the set of rational functions of the form ut^m, where u is a unit, t is a local parameter and $m \geq -1$. If the image of t in $\mathcal{O}_{X,p}$ is s^a, where s is a local parameter in $\mathcal{O}_{X,p}$, then the image of ut^m is us^{am}, so that $\mathcal{O}_{X,p}(f^*q)$ corresponds to rational functions with a pole no worse than a at p. It is easy to check that the LHS corresponds to the same rational functions.

(c) Again by linearity, we just have to check what happens to $\mathcal{O}_{\mathbb{P}^n_k}(1)$. Let $H \subset \mathbb{P}^n_k$ be a hyperplane which does not contain X. This corresponds to a linear function L. Then the restriction l of L to X is a global section of $\mathcal{O}_X(1)$, whose zero locus is the Cartier divisor

$Y = H \cap X$. The underlying Weil divisor is clearly the one defined in (II.6.2).
6. (i) Since tensor product commutes with i^*, we might as well assume that $X = C$ is a smooth projective curve. Suppose that L and M are two line bundles on C. Then we may find Cartier divisors D and E such that $L = \mathcal{O}_C(D)$ and $M = \mathcal{O}_C(E)$. In this case

$$L \underset{\mathcal{O}_C}{\otimes} M = \mathcal{O}_C(D + E).$$

Finally, clearly the degree of $D+E$ is the sum of the degrees of D and E.
(ii) Using (i), we may replace L by $L^{\otimes m}$ and so we might as well assume that L is base point free. Suppose that $C \subset X$ is a projective curve and let $i\colon C' \longrightarrow X$ be the composition of the normalisation and inclusion. Then i^*L is base point free. So we might as well assume that $X = C$ is a smooth projective curve. Pick a global section s of L. Then $D = (s) \geq 0$ and the degree of L is the degree of D which is surely non-negative.
(iii) By (i) we may assume that L is very ample, so that $L = \mathcal{O}_X(1)$, where $X \subset \mathbb{P}^n_K$. Suppose that $C \subset X$ is a projective curve. Then $\mathcal{O}_X(1) \cdot C = \mathcal{O}_{\mathbb{P}^n_K}(1) \cdot C$. So we might as well assume that $X = \mathbb{P}^n_K$. In this case, zeroes of sections of $\mathcal{O}_{\mathbb{P}^n_K}(1)$ correspond to hyperplanes in $\mathbb{P}^n$. Given C, pick a hyperplane H which does not contain C. Then H intersects C and so $D = H \cap C$ is a non-zero effective divisor on C which represents $L|_C$. As $D \geq 0$ is non-zero the degree of D is positive.
(iv) We start with a general observation. Let C be a smooth projective curve and let L be a line bundle of degree zero. If $s \in H^0(C, L)$ is a global section and $D = (s)$ then $D \geq 0$ and D has degree zero. It follows that $D = 0$ and so $L \simeq \mathcal{O}_C$. So a line bundle of degree zero is base point free if and only if it is trivial, that is, isomorphic to $\mathcal{O}_E$.
Now if L has degree zero then so does $L^{\otimes m}$. So if L has degree zero then some power of L is base point free if and only if L is torsion, that is, some power of L is the trivial line bundle. We now calculate an explicit example (in fact any smooth projective curve C over $\mathbb{C}$ has plenty of line bundles of degree zero which are not torsion (unless $C = \mathbb{P}^1$)).
Let $E \subset \mathbb{P}^2_k$ be the smooth cubic $Y^2Z = X^3 - XZ^2$. Note that $P_0 = [0 : 1 : 0]$ is a point of E. The map

$$f\colon E \longrightarrow \mathrm{Pic}(E)$$

which sends P to $\mathcal{O}_E(P - P_0)$ sets up a correspondence between points of E and line bundles of degree zero.
Now E is an algebraic group, where P_0 is the identity point and addition is given by the rule that three points sum to zero if and only if they

are collinear. With this group structure, f is a group homomorphism. All of this is observed in (II.6.10.2).

So all we want to do is find a point P of E which is not torsion. Since the set of closed points is uncountable (it has the same cardinality as $\mathbb{C}$), it suffices to show that the set of torsion points is countable. Fix a positive integer d. It suffices to show that there are only finitely many points of order d. Since E is an algebraic group the set of points of order dividing d is a Zariski closed subset. So all we have to show is that there is a point whose order does not divide d.

Now if there are infinitely many points of order d then there are infinitely many points of order some prime p dividing d. So all we have to show is that not every point other than P_0 has order a prime p. There are many ways to do this; here is one.

Consider first the case $p = 2$. A point P has order 2 if and only if the tangent line to P meets E at P_0 (the tangent line to P_0 is given by $Z = 0$ and this meets E only at the point P_0; P_0 is a flex point and $Z = 0$ is a flex line). The set of lines through P_0 is given by $aX + bZ = 0$ where $[a : b] \in \mathbb{P}^1$. Ignoring the line $Z = 0$ and working the affine plane $Z \neq 0$, are looking at lines of the form $x = c$, where c is a contant. Plugging this into the equation $y^2 = x^3 - x$ of the cubic, we have

$$y^2 = c^3 - c,$$

and we want to know when this has only one solution. In this case $c^3 - c = 0$, so that $c = 0$, $c = 1$ and $c = -1$ are the only possibilities. This gives us the points $P_1 = [0 : 0 : 1]$, $P_2 = [1 : 0 : 1]$ and $P_3 = [-1 : 0 : 1]$. Thus there are points of order two but not every point has order two (or is the identity).

Since there are points of order 2 there are only finitely many points of order d, where d is odd. But then there are countably many torsion points and there are points of infinite order. These correspond to line bundles of degree zero no multiple of which is base point free.

It seems worthwhile to consider this story from the viewpoint of complex manifolds and Riemann surfaces. In this case E is isomorphic to the quotient of $\mathbb{C}$ modulo a lattice Λ and the group structure descends from $\mathbb{C}$. Points of order d correspond to elements of $\frac{1}{d}\Lambda$ modulo elements of Λ. For any fixed d the set of such points of $\mathbb{Z}_d \times \mathbb{Z}_d$ (as a Lie group E is nothing more than a product of two circles). So there are plenty of points of infinite order.

There are other examples we could have chosen. We could have started with the cuspidal cubic $Y^2Z = X^3$. In this case line bundles of degree zero correspond to points of the cuspidal cubic other than the origin;

the underlying group is $\mathbb{G}_a$ and this has plenty of points which are not torsion (at least if the ground field is not of characteristic p).
If we start with the nodal cubic, then the group line bundles of degree zero is isomorphic to $\mathbb{G}_m$. There are plenty of elements of this group which are not torsion provided the ground field is not the algebraic closure of a finite field.