MODEL ANSWERS TO HWK #2

1. It suffices to check that the map is an isomorphism on stalks. Suppose that $x \in X$. By assumption there are open neighbourhoods U and V of x and isomorphisms $\mathcal{L}|_U \simeq \mathcal{O}_U$, $\mathcal{M}|_V \simeq \mathcal{O}_V$. Passing to the open subset $U \cap V$ we may as well assume that $\mathcal{L} = \mathcal{M} = \mathcal{O}_X$.
Let $A = \mathcal{O}_{X,x}$. Then A is a local ring and f induces a surjective A-module homomorphism $\phi\colon A \longrightarrow A$. ϕ is given by multiplication by an element a of A. Suppose that $\phi(b) = 1$. Then $ab = 1$ and so a is a unit and ϕ is an isomorphism. Thus f is an isomorphism on stalks and f is an isomorphism.
2. The vector space W of polynomials of degree $m + n - 1$ has dimensional $m + n$ and the polynomials $x^i f$, $0 \leq i \leq n - 1$ and $x^j g$, $0 \leq j \leq m - 1$ are $m + n$ elements of this vector space. So these polynomials are dependent if and only if they don't span. If f and g have a common root, then any linear combination of $x^i f$, $0 \leq i \leq n-1$ and $x^j g$, $0 \leq j \leq m - 1$ has the same root, in which case they don't span, so that they must be dependent.
Suppose that they are dependent. The polynomials $x^i f$, $0 \leq i \leq n - 1$ span a vector subspace U of dimension n and the polynomials $x^j g$, $0 \leq j \leq m - 1$, span a vector subspace V of dimension m. If the polynomials are dependent then U and V intersect non-trivially, so that U and V contain a non-zero polynomial h of degree at most $m+n-1$. The elements of U are multiples of f and the elements of V are multiples of g so that h is a multiple of f and g. But then f and g must have a common factor. As K is a algebraically closed it follows that f and g have a common root.
Putting all of this together we see that f and g have a common root if and only if the polynomials $x^i f$, $0 \leq i \leq n - 1$ and $x^j g$, $0 \leq j \leq m - 1$ are dependent. Therefore we can take the resultant be the following

determinant:

$$\begin{vmatrix}
a_0 & a_1 & a_2 & \dots & a_m & 0 & 0 & \dots & 0 \\
0 & a_0 & a_1 & \dots & a_{m-1} & a_m & 0 & \dots & 0 \\
0 & 0 & a_0 & \dots & a_{m-2} & a_{m-1} & a_m & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & a_0 & a_1 & a_2 & \dots & a_m \\
b_0 & b_1 & b_2 & \dots & b_m & b_{m+1} & b_{m+2} & \dots & 0 \\
0 & b_0 & b_1 & \dots & b_{m-1} & b_m & b_{m+1} & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & b_{n-m} & b_{n-m+1} & b_{n-m+2} & \dots & b_n
\end{vmatrix}.$$

3. (i) Pick a basis $v_1, v_2, \dots, v_n$ of V. Then ϕ is represented by a $n \times n$ matrix $A \in \mathbb{A}^{n^2}_K$, which is an irreducible affine variety.
(ii) We are certainly free to enlarge K. So we may assume that K is algebraically closed. Let $U \subset \mathbb{A}^{n^2}_K$ be the subset consisting of matrices with n distinct eigenvalues. If $f(x) = \det(A - xI)$ is the characteristic polynomial then the eigenvalues are roots of this polynomial and the locus U is the locus where the $f(x)$ has no repeated roots. Now the function which takes a matrix and assigns the coefficients of the characteristic polynomial is naturally a morphism

$$c\colon \mathbb{A}^{n^2}_K \longrightarrow \mathbb{A}^{n+1}_K.$$

The function which takes a polynomial f and assigns the value of $R(f, f')$ is also a morphism

$$r\colon \mathbb{A}^{n+1}_K \longrightarrow K.$$

The composition is a morphism

$$r \circ c\colon \mathbb{A}^{n^2}_K \longrightarrow K,$$

which is non-zero if and only if A belongs to U. Thus U is an open subset and it is certainly non-empty; just take any diagonal matrix with non-zero distinct entries on the diagonal (K is infinite as it is algebraically closed).
The characteristic polynomial vanishes on U, which is a dense open subset and so the characteristic polynomial vanishes on $\mathbb{A}^{n^2}_K$.
4. (i) $U_{\sigma_1} = \mathbb{A}^1_K \times \mathbb{G}_m$ and $U_{\sigma_2} = \mathbb{A}^1_K \times \mathbb{G}_m$. The union is $\mathbb{A}^2_K - \{0\}$; the orbits are $\mathbb{G}^2_m = U_0$, $\mathbb{G}_m = U_{\sigma_1}$, and $\mathbb{G}_m = U_{\sigma_2}$.
(ii) Number the three maximal cones σ_1, σ_2 and σ_3 in the usual way. The primitive generators of σ_1 are e_1 and e_2, which generate the lattice and the primitive generators of σ_3 are e_1 and $-2e_1 - e_2$, which also generate the lattice. Thus $U_{\sigma_1} = U_{\sigma_1} = \mathbb{A}^2_K$. On the other hand, σ_2

is generated by e_2 and $-2e_1 - e_2$ which don't generate the lattice. In fact, we get the quadric cone:

$$U_{\sigma_2} = \operatorname{Spec} \frac{K[u,v,w]}{\langle v^2 - uw \rangle}.$$

We guess that the X is the quadric cone in $\mathbb{P}^3$:

$$Q = \operatorname{Proj} \frac{K[U,V,W,T]}{\langle V^2 - UW \rangle}.$$

When $T \neq 0$ we at least get an affine variety isomorphic to U_{σ_2}.
To check our guess, we first calculate the three affine coordinate rings of the toric variety:

$$A_{\sigma_1} = K[x,y] \quad A_{\sigma_2} = K[x^{-1}, x^{-1}y, x^{-1}y^2] \quad \text{and} \quad A_{\sigma_3} = K[xy^{-2}, y^{-1}].$$

We already have one affine open subset of Q and we want two other open affine subsets. Now if we take the standard open affine cover of $\mathbb{P}^3_K$, the only points omitted by taking three of the four open subsets are the torus invariant points $[1:0:0:0]$, $[0:1:0:0]$, $[0:0:1:0]$ and $[0:0:0:1]$. Of these the quadric cone only avoids the point $[0:1:0:0]$. So the obvious choice is to take three open subsets of Q, given by $U \neq 0$, $W \neq 0$ and $T \neq 0$, with coordinate rings:

$$\begin{aligned} B_1 &= K[V/U, T/U] \\ B_2 &= K[U/T, V/T, W/T]/\langle (U/T)(W/T) - (V/T)^2 \rangle \\ B_3 &= K[V/W, T/W]. \end{aligned}$$

Here we used the fact that $W/U = (V/U)^2$ and $U/W = (V/W)^2$.
Now we check the patching conditions. We put $y = V/U$, $x = T/U$. This matches up B_1 and A_{σ_1}. As $(V/U)(V/W) = 1$, we have $V/W = y^{-1}$ and

$$xy^{-2} = \frac{TU^2}{UV^2} = \frac{TU}{V^2} = \frac{T}{W}.$$

This matches B_3 and A_{σ_3}. Finally,

$$x^{-1} = \frac{U}{T} \qquad x^{-1}y = \frac{U}{T}\frac{V}{U} = \frac{V}{T} \qquad \text{and} \qquad x^{-1}y = \frac{U}{T}\frac{V^2}{U^2} = \frac{W}{T}.$$

so that B_2 and A_{σ_2} also match up.
(iii) Let $u_i \in M$ be the linear form which is zero on v_i and v_{i+1} and takes the value 1 on v_{i+2} (take subscripts modulo 4). Then u_1, u_2, u_3 and u_4 generate the dual cone $\check{\sigma} \subset M_{\mathbb{R}} = \mathbb{R}^3$ to σ, any three generate M and we have the relation $u_1 + u_3 = u_2 + u_4$. So

$$A_\sigma = \frac{K[a,b,c,d]}{\langle ac - bd \rangle}.$$

It follows that U_σ is the quadric cone in $\mathbb{A}^4_K$.
5. (i) As σ is a rational polyhderal cone, if σ lives in a codimension k linear subspace, then in fact it lives in a codimension k linear subspace V_2, spanned by elements of N. Choose a complement V_1 of dimension k, spanned by elements of N, so that $N_\mathbb{R} = V_1 + V_2$ and $V_1 \cap V_2 = \{0\}$. Let $N_1 = V_1 \cap N$ and $N_2 = V_2 \cap N$ so that $N = N_1 + N_2$. Let $\tau \subset V_2$ be the cone corresponding to σ. The decomposition $N = N_1 + N_2$ induces a dual decomposition $M = M_1 + M_2$, where $M_i = \mathrm{Hom}(N_i, \mathbb{Z})$ and $\check{\sigma} = \check{\tau} + (M_1)_\mathbb{R}$. It follows that $S_\sigma = S_\tau + M_1$, so that $A_\sigma = A_\tau[x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_k, x_k^{-1}]$. But then $U_\sigma = U_\tau \times \mathbb{G}_m^k$.
(ii) If Y is a variety, note that $Y \times \mathbb{G}_m$ is regular if and only if Y is regular. So, using (i), we might as well assume that σ spans $N_\mathbb{R} = \mathbb{R}^n$. In this case, it is clear that (2) and (3) are equivalent, since any set of n generators of a lattice are equivalent modulo the action of $\mathrm{GL}(n, \mathbb{Z})$. As $\mathbb{A}^n_K$ is regular, (3) implies (1).
The key is to show that (1) implies (2). Suppose that U_σ is regular. Let $\mathfrak{m} \triangleleft A_\sigma$ be the maximal ideal of x_σ. Note that $\mathfrak{m}$ is a direct sum of eigenspaces, spanned by χ^u, where $u \in S_\sigma$. Further, $\mathfrak{m}^2$ is generated by elements of the form χ^{u+v}, where u and v are non-zero elements of S_σ. So a basis for $\mathfrak{m}/\mathfrak{m}^2$ is given by χ^u, where u ranges over the elements of S_σ that are not the sum of two non-zero elements S_σ. As U_σ is regular, $\mathfrak{m}/\mathfrak{m}^2$ is an n-dimensional K-vector space. Let $w_1, w_2, \dots, w_m$ be the primitive generators of the edges of $\check{\sigma}$. Then w_i is not a sum of two other elements of $\check{\sigma}$. It follows $m \leq n$, whence $m = n$, since $\check{\sigma}$ spans $M_\mathbb{R}$ and it is strongly convex (as σ spans $N_\mathbb{R}$).
Let $\sigma' \subset \check{\sigma}$ be the face spanned by $w_1, w_2, \dots, w_{n-1}$. Any element belonging to $S_\sigma \cap \sigma'$ is a sum of $w_1, w_2, \dots, w_{n-1}$, so that by induction we may assume that $w_1, w_2, \dots, w_{n-1}$ span the lattice spanned by σ'. Changing coordinates, we may assume that $w_i = e_i$, $1 \leq i \leq n-1$. Possibly reflecting in the plane spanned by $e_1, e_2, \dots, e_{n-1}$, we may assume that the last coordinate of w_n is positive. In this case $\check{\sigma}$ contains a vector of the form $(a_1, a_2, \dots, a_{n-1}, 1) \in \mathbb{N}^n$ (take a_i large). Thus the elements of S_σ generate the lattice N, in which case $w_1, w_2, \dots, w_n$ generate the lattice M. In this case σ is also spanned by vectors $v_1, v_2, \dots, v_n$ which generate the lattice N. Thus (1) implies (2).