MODEL ANSWERS TO HWK #1

1. Let U be the free abelian semigroup generated by $v_1, v_2, \dots, v_m$ (so that U is abstractly isomorphic to $\mathbb{N}^m$). Define a semigroup homomorphism $U \longrightarrow S_\sigma$ by sending v_i to u_i. This is surjective and the kernel is generated by relations of the form

$$\sum a_i v_i - \sum b_i v_i,$$

where

$$\sum a_i u_i = \sum b_i u_i$$

in S_σ. The group algebra A_σ is generated by $x_i = \chi^{u_i}$. Define a ring homomorphism

$$K[x_1, x_2, \dots, x_n] \longrightarrow A_\sigma,$$

by sending x_i to χ^{u_i}. Then the kernel certainly contains relations of the form

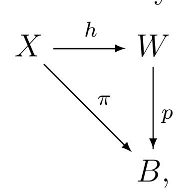
$$x_1^{a_1} x_2^{a_2} \dots x_m^{a_m} - x_1^{b_1} x_2^{b_2} \dots x_m^{b_m},$$

where

$$\sum a_i u_i = \sum b_i u_i.$$

If we quotient out by these relations, then we get a vector space Q with one dimensional eigenspaces indexed by $u \in S_\sigma$. As A_σ has the same property (the corresponding eigenspaces are spanned by the monomials χ^u) the induced linear map $Q \longrightarrow A_\sigma$ is an isomorphism. So the given relations actually generate the kernel.

2. We are free to replace Y by something bigger; so we may assume that Y is projective. Let $W \subset Y \times B$ be the closure of the image of X under the morphism $f \times \pi$. Then we may factor π into two morphisms,



where p is restriction of the second projection. Note that the second morphism is automatically projective and the first morphism is projective as the composition is projective and the second morphism is separated.

By assumption $h(\pi^{-1}(b_0))$ is a point w_0 in W. But w_0 is then the fibre of p over b_0. By upper semi-continuity of the dimensions of a fibre, it follows that there is an open subset U of B, such that $p^{-1}(b)$ is zero

dimensional, for every $b \in U$. In this case, the dimension of the fibres of h over $p^{-1}(U)$ is at least n, whence the dimension of any fibre of h is at least n.
Pick $w \in W$. Then the fibre $h^{-1}(w)$ has dimension at least n. On the other hand, $h^{-1}(w) \subset \pi^{-1}(p(w))$, which has dimension n, so that $h^{-1}(w)$ is a union of some of the irreducible components of $\pi^{-1}(p(w))$. It follows that $h(\pi^{-1}(p(w))) = p^{-1}(p(w))$ is a finite set of points. As $\pi^{-1}(p(w))$ is connected, it follows that the image is a point.
3. Let $\pi\colon A \times A \longrightarrow A$ be projection onto the first factor and let $f\colon A \times A \longrightarrow A$ be the morphism which sends (g, h) to ghg^{-1}. Then $\pi^{-1}(e) = \{e\} \times A$ is sent to a point by f. As the fibres of π are irreducible of the same dimension and π is surjective, it follows that if $a \in A$ then f sends $\{a\} \times A$ to a point. As f sends (a, e) to e it follows that $aba^{-1} = e$, so that A is commutative.
4. It suffices to prove that if π sends the identity to the identity then π is a group homomomorphism. Consider the morphism of projective varieties

$$f\colon A \times A \longrightarrow B,$$

which sends (a_1, a_2) to $\pi(a_1 + a_2) - \pi(a_1) - \pi(a_2)$. Let $\phi\colon A \times A \longrightarrow A$ denote projection onto the first factor. Then f sends $\phi^{-1}(e)$ to the identity of B, where e is the identity of A. By the rigidity lemma f sends $\{a\} \times A$ to a point. But $f(a, e)$ is the identity so $f(a_1, a_2)$ is the identity of B, for every a_1 and $a_2 \in A$. But then π is a group homomomorphism.
5. We may suppose that π sends one to one and we need to prove that π is a group homomorphism in this case. Since $\mathbb{G}_m^n$ is a product in the category of varieties and algebraic groups, it suffices to prove this result when $H = \mathbb{G}_m$. At the level of coordinate rings, we have a ring homomorphism

$$\phi\colon K[\mathbb{Z}] \longrightarrow K[\mathbb{Z}^m].$$

ϕ is determined by where we send t, which is a polynomial in $x_1, x_2, \dots, x_m$ and their inverses. Since this polynomial has no zeroes or poles, it must be a scalar multiple of a monomial (possibly with negative powers). As 1 is sent to 1 this scalar is 1. But map

$$(x_1, x_2, \dots, x_m) \longrightarrow M,$$

where $M = x_1^{a_1} x_2^{a_2} \dots x_m^{a_m}$ is clearly a group homomorphism.
6. We first show that f is a morphism. One can use the valuative criteria but it is more straightforward to prove this result directly. It suffices to prove that if we are given a rational map

$$f\colon \mathbb{A}^1 \dashrightarrow \mathbb{P}^n,$$

then f is defined at the origin. Using the local description of morphisms, we have

$$t \longrightarrow [f_0 : f_1 : \cdots : f_n],$$

where $f_i = g_i/h_i$ is a rational function. Let $m_i = \nu(f_i)$, where ν measures the multiplicity of f_i at the origin. Let $m = \min m_i$. Then f is equally well represented by

$$t \longrightarrow [f'_0 : f'_1 : \cdots : f'_n],$$

where $f'_i = t^m f_i$. By our choice of m, f'_i does not have a pole at 0 and at least one f'_i is non-zero at 0. Thus f is a morphism.

We may assume that $f(0)$ is the identity of A. As $\mathbb{P}^1 - \{\infty\} \simeq \mathbb{G}_a$ it follows that $f(a+b) = f(a) + f(b)$, for all a and $b \in \mathbb{P}^1 - \{\infty\}$. As $\mathbb{P}^1 - \{0, \infty\} \simeq \mathbb{G}_m$ it follows that $f = \tau_p \circ g$, where $g(1)$ is the identity. In this case $g(ab) = g(a) + g(b)$ and so

$$f(ab) - p = g(ab) = g(a) + g(b) = f(a) + f(b) - 2p,$$

that is

$$f(ab) + p = f(a) + f(b) = f(a+b).$$

This is clearly absurd, unless $f(a)$ is the identity of A, for every $a \in \mathbb{P}^1$.

Now suppose that the groundfield is $\mathbb{C}$. Then A is a complex torus, the quotient of $\mathbb{C}^n$ by a lattice Λ of rank $2n$ and $\mathbb{P}^1$ is the Riemann sphere. The universal cover of A is $\mathbb{C}^n$ and the universal cover of $\mathbb{P}^1$ is the Riemann sphere. By the universal property of the universal cover, there is an induced commutative diagram

$$\begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{g} & \mathbb{C}^n \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \xrightarrow{f} & A. \end{array}$$

If g is not constant then one of the induced holomorphic maps

$$\mathbb{P}^1 \longrightarrow \mathbb{C},$$

given by projection, is not constant. By the open mapping theorem the image is open; as $\mathbb{P}^1$ is compact the image is compact, whence closed. The only open and closed subset of $\mathbb{C}$ is $\mathbb{C}$ itself, but this is not compact, a contradiction. Hence g is constant and so f is constant as well.