9. Algebraic versus analytic geometry

An analytic variety is defined in a very similar way to a scheme. First of all, given an open subset of $U \subset \mathbb{C}^n$, we say $X \subset U$ is an analytic closed subset if locally X is defined by the vanishing of holomorphic (equivalently analytic functions). A regular analytic function on X is then something which is the restriction of a holomorphic (equivalently analytic) functions from $U \subset \mathbb{C}^n$, so that the ring of regular functions on X is

$$\frac{\mathcal{O}_U^{\mathrm{an}}(U)}{I},$$

where $\mathcal{O}_U^{\mathrm{an}}(U)$ is the ring of holomorphic functions on U.

Globally, we have a locally ringed space $(X, \mathcal{O}_X^{\mathrm{an}})$, where X is locally isomorphic to an analytic closed subset of some open subset $U \subset \mathbb{C}^n$ together with its sheaf of analytic functions.

Theorem 9.1 (Chow's Theorem)**.** *Let $X \subset \mathbb{P}^n$ be a closed analytic subset of projective space.*

Then X is a projective subscheme.

More generally, given a (n algebraic) scheme $(X, \mathcal{O}_X)$ of finite type over $\mathbb{C}$, we can construct an analytic variety $(X^{\mathrm{an}}, \mathcal{O}_X^{\mathrm{an}})$ in a fairly obvious way. To get X^{an} we just have to ditch the points which are not closed and enrich the topology.

The resulting functor, from the category of schemes of finite type over $\mathbb{C}$ to the category of analytic spaces, induces an equivalence of categories between projective schemes and compact analytic subschemes of projective space. The key point is that a morphism of schemes or analytic spaces is represented by the graph; the graph sits inside the product so that if the domain and range are projective then so is the graph and then one just applies (9.1).

Note that if we drop the condition that $X \subset \mathbb{P}^n$ is an analytic closed subset then there is no longer an equivalence of categories. For example $\mathbb{C}$ has lots of holomorphic functions which are nowhere near algebraic.

If X is an analytic space whose local rings are all regular then X is locally modeled on open subsets of $\mathbb{C}^n$, so that X is a complex manifold.

A basic result in the theory of C^∞-maps is Sard's Theorem, which states that the set of points where a map is singular is a subset of measure zero (of the base). Since any holomorphic map between complex manifolds is automatically C^∞, and the derivative of a polynomial is the same as the derivative as a holomorphic function, it follows that any morphism between varieties, over $\mathbb{C}$, is smooth over an open subset. In fact by the Lefschetz principle, this result extends to any variety over $\mathbb{C}$.

Theorem 9.2. *Let $f\colon X \longrightarrow Y$ be a morphism of varieties over a field of characteristic zero.*

Then there is a dense open subset U of Y such that if $q \in U$ and $p \in f^{-1}(q) \cap X_{sm}$ then the differential $df_p\colon T_pX \longrightarrow T_qY$ is surjective. Further, if X is smooth, then the fibres $f^{-1}(q)$ are smooth if $q \in U$.

Let us recall the Lefschetz principle. First recall the notion of a first order theory of logic. Basically this means that one describes a theory of mathematics using a theory based on predicate calculus. For example, the following is a true statement from the first order theory of number theory,

$$\forall n \forall x \forall y \forall z \quad n \geq 3 \implies x^n + y^n \neq z^n.$$

One basic and desirable property of a first order theory of logic is that it is complete. In other words every possible statement (meaning anything that is well-formed) can be either proved or disproved. It is a very well-known result that the first order theory of number theory is not complete (Gödel's Incompleteness Theorem). What is perhaps more surprising is that there are interesting theories which are complete.

Theorem 9.3. *The first order logic of algebraically closed fields of characteristic zero is complete.*

Notice that a typical statement of the first order logic of fields is that a system of polynomial equations does or does not have solution. Since most statements in algebraic geometry turn on whether or not a system of polynomial equations have a solution, the following result is very useful.

Principle 9.4 (Lefschetz Principle)**.** *Every statement in the first order logic of algebraically closed fields of characteristic zero, which is true over $\mathbb{C}$, is in fact true over any algebraically closed field of characteristic.*

In fact this principle is immediate from (9.3). Suppose that p is a statement in the first order logic of algebraically closed fields of characteristic zero. By completeness, we can either prove p or not p. Since p holds over the complex numbers, there is no way we can prove not p. Therefore there must be a proof of p. But this proof is valid over any field of characteristic zero, so p holds over any algebraically closed field of characteristic zero.

A typical application of the Lefschetz principle is (9.2). By Sard's Theorem, we know that (9.2) holds over $\mathbb{C}$. On the other hand, (9.2), can be reformulated in the first order logic of algebraically closed fields of characteristic zero. Therefore by the Lefschetz principle, (9.2) is true over algebraically closed field of characteristic zero.

Perhaps even more interesting, is that (9.2) fails in characteristic p. Let $f\colon \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ be the morphism $t \longrightarrow t^p$. If we fix s, then the equation

$$x^p - s$$

is purely inseparable, that is, has only one root. Thus f is a bijection. However, df is the zero map, since $dz^p = pz^{p-1}dz = 0$. Thus df_p is nowhere surjective. Note that the fibres of this map, as schemes, are isomorphic to zero dimensional schemes of length p.