8. Kähler differentials

Let A be a ring, B an A-algebra and M a B-module.

Definition 8.1. *An A-**derivation** of B into M is a map $\mathrm{d}\colon B \longrightarrow M$ such that*

(1) $\mathrm{d}(b_1 + b_2) = \mathrm{d}b_1 + \mathrm{d}b_2$.
(2) $\mathrm{d}(bb') = b'\mathrm{d}b + b\mathrm{d}b'$.
(3) $\mathrm{d}a = 0$.

Definition 8.2. *The module of **relative differentials**, denoted $\Omega_{B/A}$, is a B-module together with an A-derivation, $\mathrm{d}\colon B \longrightarrow \Omega_{B/A}$, which is universal with this property:*

If M is a B-module and $\mathrm{d}'\colon B \longrightarrow M$ is an A-derivation then there exists a unique B-module homomorphism $f\colon \Omega_{B/A} \longrightarrow M$ which makes the following diagram commute:

$$\begin{array}{ccc} B & \xrightarrow{d} & \Omega_{B/A} \\ & {\scriptstyle d'}\searrow & \downarrow{\scriptstyle f} \\ & & M. \end{array}$$

One can construct the module of relative differentials in the usual way; take the free B-module, with generators

$$\{\, \mathrm{d}b \mid b \in B \,\},$$

and quotient out by the three obvious sets of relations

(1) $\mathrm{d}(b_1 + b_2) - \mathrm{d}b_1 - \mathrm{d}b_2$,
(2) $\mathrm{d}(bb') - b'\mathrm{d}b - b\mathrm{d}b'$, and
(3) $\mathrm{d}a$.

The map $\mathrm{d}\colon B \longrightarrow M$ is the obvious one.

Example 8.3. *Let $B = A[x_1, x_2, \dots, x_n]$. Then $\Omega_{B/A}$ is the free B-module generated by $\mathrm{d}x_1, \mathrm{d}x_2, \dots, \mathrm{d}x_n$.*

Proposition 8.4. *Let A' and B be A-algebras and $B' = B \underset{A}{\otimes} A'$. Then*

$$\Omega_{B'/A'} = \Omega_{B'/A} \underset{B}{\otimes} B'$$

Furthermore, if S is a multiplicative system in B, then

$$\Omega_{S^{-1}B/A} = S^{-1}\Omega_{B/A}.$$

Suppose that $X = \operatorname{Spec} B \longrightarrow Y = \operatorname{Spec} A$ is a morphism of affine schemes. The **sheaf of relative differentials** $\Omega_{X/Y}$ is the quasi-coherent sheaf associated to the module of relative differentials $\Omega_{B/A}$.

Example 8.5. *Let* $X = \operatorname{Spec} \mathbb{R}$ *and* $Y = \operatorname{Spec} \mathbb{Q}$*. Then* $\mathrm{d}\pi \in \Omega_{X/Y} = \Omega_{\mathbb{R}/\mathbb{Q}}$ *is a non-zero differential.*

One could use the affine case to construct the sheaf of relative differentials globally. A better way to proceed is to use a little bit more algebra (and geometric intuition):

Proposition 8.6. *Let* B *be an* A*-algebra. Let*

$$B \underset{A}{\otimes} B \longrightarrow B,$$

be the diagonal morphism $b \otimes b' \longrightarrow bb'$ *and let* I *be the kernel. Consider* $B \underset{A}{\otimes} B$ *as a* B*-module by multiplication on the left. Then* I/I^2 *inherits the structure of a* B*-module. Define a map*

$$\mathrm{d} \colon B \longrightarrow \frac{I}{I^2},$$

by the rule

$$\mathrm{d}b = 1 \otimes b - b \otimes 1.$$

Then I/I^2 *is the module of differentials.*

Now suppose we are given a morphism of schemes $f \colon X \longrightarrow Y$. This induces the diagonal morphism

$$\Delta \colon X \longrightarrow X \underset{Y}{\times} X.$$

Then Δ defines an isomorphism of X with its image $\Delta(X)$ and this is locally closed in $X \underset{Y}{\times} X$, that is, there is an open subset $W \subset X \underset{Y}{\times} X$ and $\Delta(X)$ is a closed subset of W (it is closed in $X \underset{Y}{\times} X$ if and only if X is separated).

Definition 8.7. *Let* $\mathcal{I}$ *be the sheaf of ideals of* $\Delta(X)$ *inside* W*. The* ***sheaf of relative differentials***

$$\Omega_{X/Y} = \Delta^* \left(\frac{\mathcal{I}}{\mathcal{I}^2} \right).$$

Theorem 8.8 (Euler sequence)**.** *Let* A *be a ring, let* $Y = \operatorname{Spec} A$ *and* $X = \mathbb{P}^n_A$.

Then there is a short exact sequence of $\mathcal{O}_X$*-modules*

$$0 \longrightarrow \Omega_{X/Y} \longrightarrow \mathcal{O}_X(-1)^{n+1} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Proof. Let $S = A[x_0, x_1, \dots, x_n]$ be the homogeneous coordinate ring of X. Let E be the graded S-module $S(-1)^{n+1}$, with basis $e_0, e_1, \dots, e_n$ in degree one. Define a (degree 0) homomorphism of graded S-modules

$E \longrightarrow S$ by sending $e_i \longrightarrow x_i$ and let M be the kernel. We have a left exact sequence

$$0 \longrightarrow M \longrightarrow E \longrightarrow S.$$

This gives rise to a short exact sequence of $\mathcal{O}_X$-modules,

$$0 \longrightarrow \tilde{M} \longrightarrow \mathcal{O}_X(-1)^{n+1} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Note that even though $E \longrightarrow S$ is not surjective, it is surjective in all non-negative degrees, so that the map of sheaves is surjective.

It remains to show that $\tilde{M} \simeq \Omega_{X/Y}$. First note that if we localise at x_i, then $E_{x_i} \longrightarrow S_{x_i}$ is a surjective homomorphism of free S_{x_i}-modules, so that M_{x_i} is a free S_{x_i}-module of rank n, generated by

$$\{ e_j - \frac{x_j}{x_i} e_i \,|\, j \neq i \}.$$

It follows that if U_i is the standard open affine subset of X defined by x_i then $\tilde{M}|_{U_i}$ is a free $\mathcal{O}_{U_i}$-module of rank n generated by the sections

$$\{ \frac{1}{x_i} e_j - \frac{x_j}{x_i^2} e_i \,|\, j \neq i \}.$$

(We need the extra factor of x_i to get elements of degree zero.)

We define a map

$$\phi_i \colon \Omega_{X/Y}|_{U_i} \longrightarrow \tilde{M}|_{U_i},$$

as follows. As $U_i = \operatorname{Spec} k[\frac{x_0}{x_i}, \frac{x_1}{x_i}, \dots, \frac{x_n}{x_i}]$, it follows that $\Omega_{X/Y}$ is the free $\mathcal{O}_{U_i}$-module generated by

$$\mathrm{d}\left(\frac{x_0}{x_i}\right), \mathrm{d}\left(\frac{x_1}{x_i}\right), \dots, \mathrm{d}\left(\frac{x_n}{x_i}\right).$$

So we define ϕ_i by the rule

$$\mathrm{d}\left(\frac{x_j}{x_i}\right) \longrightarrow \frac{1}{x_i} e_j - \frac{x_j}{x_i^2} e_i.$$

ϕ_i is clearly an isomorphism. We check that we can glue these maps to a global isomorphism,

$$\phi \colon \Omega_{X/Y} \longrightarrow \tilde{M}.$$

On $U_i \cap U_j$, we have

$$\left(\frac{x_k}{x_i}\right) = \left(\frac{x_k}{x_j}\right)\left(\frac{x_j}{x_i}\right).$$

Hence in $(\Omega_{X/Y})|_{U_i \cap U_j}$ we have

$$\mathrm{d}\left(\frac{x_k}{x_i}\right) - \frac{x_k}{x_j}\mathrm{d}\left(\frac{x_j}{x_i}\right) = \frac{x_j}{x_i}\mathrm{d}\left(\frac{x_k}{x_j}\right).$$

If we apply ϕ_i to the LHS and ϕ_j to the RHS, we get the same thing, namely

$$\frac{1}{x_i x_j}(x_j e_k - x_k e_j).$$

Thus the isomorphisms ϕ_i glue together. ☐

Definition 8.9. *A variety is **smooth** (aka non-singular) if all of its local rings are regular local rings.*

Theorem 8.10. *The localisation of any regular local ring at a prime ideal is a regular local ring.*

Thus to check if a variety is smooth it is enough to consider only the closed points.

Theorem 8.11. *Let X be an irreducible separate scheme of finite type over a an algebraically closed field k.*

Then $\Omega_{X/k}$ is locally free of rank $n = \dim X$ if and only if X is a smooth variety over k.

If $X \longrightarrow Z$ is a morphism of schemes and $Y \subset X$ is a closed subscheme, with ideal sheaf $\mathcal{I}$. Then there is an exact sequence of sheaves on Z,

$$\frac{\mathcal{I}}{\mathcal{I}^2} \longrightarrow \Omega_{X/Z} \otimes \mathcal{O}_Y \longrightarrow \Omega_{Y/Z} \longrightarrow 0.$$

Theorem 8.12. *Let X be a smooth variety of dimension n. Let $Y \subset X$ be an irreducible closed subscheme with sheaf of ideals $\mathcal{I}$.*

Then Y is smooth if and only if

(1) $\Omega_{Y/k}$ is locally free, and
(2) the sequence above is also left exact:

$$0 \longrightarrow \frac{\mathcal{I}}{\mathcal{I}^2} \longrightarrow \Omega_{X/k} \otimes \mathcal{O}_Y \longrightarrow \Omega_{Y/k} \longrightarrow 0.$$

Furthermore, in this case, $\mathcal{I}$ is locally generated by $r = \operatorname{codim}(Y, X)$ elements and $\frac{\mathcal{I}}{\mathcal{I}^2}$ is locally free of rank r on Y.

Proof. Suppose (1) and (2) hold. Then $\Omega_{Y/k}$ is locally free and so we only have to check that its rank q is equal to the dimension of Y. Then $\mathcal{I}/\mathcal{I}^2$ is locally free of rank $n - q$. Nakayama's lemma implies that $\mathcal{I}$ is locally generated by $n-q$ elements and so $\dim Y \geq n-(n-q) = q$. On the other hand, if $y \in Y$ is any closed point $q = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$ and so $q \geq \dim Y$. Thus $q = \dim Y$. This also establishes the last statement.

Now suppose that Y is smooth. Then $\Omega_{Y/k}$ is locally free of rank $q = \dim Y$ and so (1) is immediate. On the other hand, there is an exact sequence

$$\frac{\mathcal{I}}{\mathcal{I}^2} \longrightarrow \Omega_{X/k} \otimes \mathcal{O}_Y \longrightarrow \Omega_{Y/k} \longrightarrow 0.$$

Pick a closed point $y \in Y$. As $\mathcal{I}/\mathcal{I}^2$ is locally free of rank $r = n - q$, we may pick sections $x_1, x_2, \dots, x_r$ of $\mathcal{I}$ such that $\mathrm{d}x_1, \mathrm{d}x_2, \dots, \mathrm{d}x_r$ generate the kernel of the second map.

Let $Y' \subset X$ be the corresponding closed subscheme. Then, by construction, $\mathrm{d}x_1, \mathrm{d}x_2, \dots, \mathrm{d}x_r$ generate a free subsheaf of rank r of $\Omega_{X/k} \otimes \mathcal{O}_{Y'}$ in a neighbourhood of y. It follows that for the exact sequence for Y'

$$\frac{\mathcal{I}'}{\mathcal{I}'^2} \longrightarrow \Omega_{X/Z} \otimes \mathcal{O}_Y \longrightarrow \Omega_{Y'/Z} \longrightarrow 0,$$

the first map is injective and $\Omega_{Y'/k}$ is locally free of rank $n - r$. But then Y' is smooth and $\dim Y' = \dim Y$. As $Y \subset Y'$ and Y' is integral, we must have $Y = Y'$ and this gives (2). □

Theorem 8.13 (Bertini's Theorem)**.** *Let $X \subset \mathbb{P}^n_k$ be a closed smooth projective variety. Then there is a hyperplane $H \subset \mathbb{P}^n_k$, not containing X, such that $H \cap X$ is regular at every point.*

Furthermore the set of such hyperplanes forms an open dense subset of the linear system $|H| \simeq \mathbb{P}^n_k$.

Proof. Let $x \in X$ be a closed point. Call a hyperplane H **bad** if either H contains X or H does not contain X but it does contain x and $X \cap H$ is not regular at x. Let B_x be the set of all bad hyperplanes at x. Fix a hyperplane H_0 not containing x, defined by $f_0 \in V = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. Define a map

$$\phi_x \colon V \longrightarrow \mathcal{O}_{X,x}/\mathfrak{m}^2,$$

as follows. Given by $f \in V$, f/f_0 is a regular function on $X - X \cap X_0$. Send f to the image of f/f_0 to its class in the quotient $\mathcal{O}_{X,x}/\mathfrak{m}^2$.. Now $x \in X \cap H$ if and only if $\phi_x(f) \in \mathfrak{m}$. Now $x \in X \cap H$ is a regular point if and only if $\phi_x(f) \neq 0$.

Thus B_x is precisely the kernel of ϕ_x. Now as k is algebraically closed and x is a closed point, ϕ_x is surjective. If $\dim X = r$ then $\mathcal{O}_{X,x}/\mathfrak{m}^2$ has dimension $r + 1$ and so B_x is a linear subspace of $|H|$ of dimension $n - r - 1$.

Let $B \subset X \times |H|$ be the set of pairs (x, H) where $H \in B_x$. Then B is a closed subset. Let $p \colon B \longrightarrow X$ and $q \colon B \longrightarrow |H|$ denote projection onto either factor. p is surjective, with irreducible fibres of dimension $n - r - 1$. It follows that B is irreducible of dimension $r + (n - r - 1) = n - 1$. The image of q has dimension at most $n - 1$. Hence $q(B)$ is a proper closed subset of $|H|$. □

Remark 8.14. *We will see later that $H \cap X$ is in fact connected, whence irreducible, so that in fact $Y = H \cap X$ is a smooth subvariety.*

Definition 8.15. *Let X be a smooth variety. The **tangent sheaf***

$$T_X = \mathbf{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X).$$

Note that the tangent sheaf is a locally free sheaf of rank equal to the dimension of X.