7. Blowing up and toric varieties

Suppose that we start with the cone σ spanned by e_1 and e_2 inside $N_{\mathbb{R}} = \mathbb{R}^2$. We have already seen that this gives the affine toric variety $\mathbb{A}^2$. Now suppose we insert the vector $e_1 + e_2$. We now get two cones σ_1 and σ_2, the first spanned by e_1 and $e_1 + e_2$ and the second spanned by $e_1 + e_2$ and e_2. Individually each is a copy of $\mathbb{A}^2$. The dual cones are spanned by f_2, $f_1 - f_2$ and f_1 and $f_2 - f_1$. So we get $\operatorname{Spec} K[y, x/y]$ and $\operatorname{Spec} K[x, x/y]$.

Suppose that we blow up $\mathbb{A}^2$ at the origin. The blow up sits inside $\mathbb{A}^2 \times \mathbb{P}^1$ with coordinates (x, y) and $[S : T]$ subject to the equations $xT = yS$. On the open subset $T \neq 0$ we have coordinates s and y and $x = sy$ so that $s = x/y$. On the open subset $S \neq 0$ we have coordinates x and t and $y = xt$ so that $t = y/x$. So the toric variety above is nothing more than the blow up of $\mathbb{A}^2$ at the origin. The central ray corresponds to the exceptional divisor E, a copy of $\mathbb{P}^1$.

A couple of definitions:

Definition 7.1. *Let G and H be algebraic groups which act on varieties X and Y. Suppose we are given an algebraic group homomorphism, $\rho\colon G \longrightarrow H$. We say that a morphism $f\colon X \longrightarrow Y$ is ρ-**equivariant** if f commutes with the action of G and H. If X and Y are toric varieties and G and H are the tori contained in X and Y then we say that f is a **toric morphism**.*

It is easy to see that the morphism defined above is toric. We can extend this picture to other toric surfaces. First a more intrinsic description of the blow up. Suppose we are given a toric surface and a two dimensional cone σ such that the primitive generators v and w of the two one dimensional faces of σ generate the lattice (so that up the action of $\mathrm{GL}(2, \mathbb{Z})$, σ is the cone spanned by e_1 and e_2). Then the blow up of the point corresponding to σ is a toric surface, which is obtained by inserting the sum $v + w$ of the two primitive generators and subdividing σ in the obvious way (somewhat like the barycentric subdivision in simplicial topology).

Example 7.2. *Suppose we start with $\mathbb{P}^2$ and the standard fan. If we insert the two vectors $-e_1$ and $-e_2$ this corresponds to blowing up two invariant points, say $[0 : 1 : 0]$ and $[0 : 0 : 1]$. Note that now $-e_1 - e_2$ is the sum of $-e_1$ and $-e_2$. So if we remove this vector this is like blowing down a copy of $\mathbb{P}^1$. The resulting fan is the fan for $\mathbb{P}^1 \times \mathbb{P}^1$.*

Note that this is an easy way to see the birational map between the quadric $Q \subset \mathbb{P}^3$ and $\mathbb{P}^2$ given by projection from a point.

Example 7.3. *Suppose we start with $\mathbb{P}^2$ and the standard fan, $v_1 = e_1$, $v_2 = e_2$ and $v_3 = -e_1 - e_2$. Suppose we insert $w_0 = e_1 + e_2$, $w_1 = -e_1$ and $w_2 = -e_2$, that is, suppose we blow up the three coordinate points.*

In the resulting fan, with six one dimensional cones, note that $v_1 = (-e_2) + (e_1 + e_2) = w_0 + w_1$, $v_2 = (e_1 + e_2) + (-e_1) = w_0 + w_2$ and $v_3 = -e_1 - e_2 = w_1 + w_2$. It follows that we may blow down the strict transform of the three lines to get another copy of $\mathbb{P}^2$, with the upside down fan w_0, w_1 and w_2.

This represents the standard Cremona transformation.

We can generalise this to higher dimensions. For example suppose we start with the standard cone for $\mathbb{A}^3$ spanned by e_1 and e_2 and e_3. If we insert the vector $e_1 + e_2 + e_3$ (thereby creating three maximal cones) this corresponds to blowing up the origin. (In fact there is a simple recipe for calculating the exceptional divisor; mod out by the central $e_1 + e_2 + e_3$; the quotient vector space is two dimensional and the three cones map to the three cones in the quotient two dimensional vector space which correspond to the fan for $\mathbb{P}^2$). Suppose we insert the vector $e_1 + e_2$. Then the exceptional locus is $\mathbb{P}^1 \times \mathbb{A}^1$. In fact this corresponds to blowing up one of the axes (the axis is a copy of $\mathbb{A}^1$ and over every point of the axis there is a copy of $\mathbb{P}^1$).

It is interesting to figure out the geometry behind the example of a toric variety which is not projective. To warm up, suppose that we start with $\mathbb{A}^3_k$. This is the toric variety associated to the fan spanned by e_1, e_2, e_3. Imagine blowing up two of the axes. This corresponds to inserting two vectors, $e_1 + e_2$ and $e_1 + e_3$. However the order in which we blow up is significant. Let's introduce some notation. If we blow up the x-axis $\pi\colon Y \longrightarrow X$ and then the y-axis, $\psi\colon Z \longrightarrow Y$, let's call the exceptional divisors E_1 and E_2, and let E_1' denote the strict transform of E_1 on Z. E_1 is a $\mathbb{P}^1$-bundle over the x-axis. The strict transform of the y-axis in Y intersects E_1 in a point p. When we blow up this curve, $E_1' \longrightarrow E_1$ blows up the point p. The fibre of E_1' over the origin therefore consists of two copies Σ_1 and Σ_2 of $\mathbb{P}^1$. Σ_1 is the strict transform of the fibre of E_1 over the origin and Σ_2 is the exceptional divisor. The fibre Σ of E_2 over the origin is a copy of $\mathbb{P}^1$. Σ and Σ_2 are the same curve in Z.

The example of a toric variety which is not projective is obtained from $\mathbb{P}^3$ by blowing up three coordinate axes, which form a triangle. The twist is that we do something different at each of the three coordinate points. Suppose that $\pi\colon X \longrightarrow \mathbb{P}^3$ is the birational morphism down to $\mathbb{P}^3$, and let E_1, E_2 and E_3 be the three exceptional divisors. Over one point we extract E_1 first then E_2, over the second point we extract first E_2 then E_3 and over the last point we extract first E_3 then E_1.

To see what has gone wrong, we need to work in the homology and cohomology groups of X. Any curve C in X determines an element of $[C] \in H_2(X, \mathbb{Z})$. Any Cartier divisor D in X determines a class $[D] \in H^2(X, \mathbb{Z})$. We can pair these two classes to get an intersection number $D \cdot C \in \mathbb{Z}$. One way to compute this number is to consider the line bundle $\mathcal{L} = \mathcal{O}_X(D)$ associated to D. Then

$$D \cdot C = \deg \mathcal{L}|_C.$$

If D is ample then this intersection number is always positive. This implies that the class of every curve is non-trivial in homology.

Suppose the reducible fibres of E_1, E_2 and E_3 over their images are $A_1 + A_2$, $B_1 + B_2$ and $C_1 + C_3$. Suppose that the general fibres are A, B and C. We suppose that A_1 is attached to B, B_1 is attached to C and C_1 is attached to A. We have

$$\begin{aligned}
[A] &= [A_1] + [A_2] \\
&= [B] + [A_2] \\
&= [B_1] + [B_2] + [A_2] \\
&= [C] + [B_2] + [A_2] \\
&= [C_1] + [C_2] + [B_2] + [A_2] \\
&= [A] + [C_2] + [B_2] + [A_2],
\end{aligned}$$

in $H_2(X, \mathbb{Z})$, so that

$$[A_2] + [B_2] + [C_2] = 0 \in H_2(X, \mathbb{Z}).$$

Suppose that D were an ample divisor on X. Then

$$0 = D \cdot ([A_2] + [B_2] + [C_2]) > D \cdot [A_2] + D \cdot [B_2] + D \cdot [C_2] > 0,$$

a contradiction.

There are a number of things to say about this way of looking at things, which lead in different directions. The first is that there is no particular reason to start with a triangle of curves. We could start with two conics intersecting transversally (so that they lie in different planes). We could even start with a nodal cubic, and just do something different over the two branches of the curve passing through the node. Neither of these examples are toric, of course. It is clear that in the first two examples, the morphism

$$\pi \colon X \longrightarrow \mathbb{P}^3,$$

is locally projective. It cannot be a projective morphism, since $\mathbb{P}^3$ is projective and the composition of projective is projective. It also follows that π is not the blow up of a coherent sheaf of ideals on $\mathbb{P}^3$. The third example is not even a variety. It is a complex manifold (and in fact it is something called an algebraic space). In particular the notion of the blow up in algebraic geometry is more delicate than it might first appear.