6. Blowing up

Let $\phi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the map

$$[X : Y : Z] \longrightarrow [YZ : XZ : XY].$$

This map is clearly a rational map. It is called a **Cremona transformation**. Note that it is a priori not defined at those points where two coordinates vanish. To get a better understanding of this map, it is convenient to rewrite it as

$$[X : Y : Z] \longrightarrow [1/X : 1/Y : 1/Z].$$

Written as such it is clear that this map is an involution, so that it is in particular a birational map.

It is interesting to check whether or not this map really is well defined at the points $[0 : 0 : 1]$, $[0 : 1 : 0]$ and $[1 : 0 : 0]$. To do this, we need to look at the closure of the graph.

To get a better picture of what is going on, consider the following map,

$$\mathbb{A}^2 \dashrightarrow \mathbb{A}^1,$$

which assigns to a point $p \in \mathbb{A}^2$ the slope of the line connecting the point p to the origin,

$$(x, y) \longrightarrow x/y.$$

Now this map is not defined along the locus where $y = 0$. Replacing $\mathbb{A}^1$ with $\mathbb{P}^1$ we get a map

$$(x, y) \longrightarrow [x : y].$$

Now the only point where this map is not defined is the origin. We consider the closure of the graph,

$$\Gamma \subset \mathbb{A}^2 \times \mathbb{P}^1.$$

Consider how Γ sits over $\mathbb{A}^2$. Outside the origin the first projection is an isomorphism. Over the origin the graph is contained in a copy of the image, that is, $\mathbb{P}^1$. Consider any line $y = tx$ through the origin. Then this line, minus the origin, is sent to the point with slope t. It follows that the closure of this line is sent to the point with slope t. Varying t, it follows that any point of the fibre over $\mathbb{P}^1$ is a point of the graph.

Thus the morphism $\pi\colon \Gamma \longrightarrow \mathbb{A}^2$ is an isomorphism outside the origin and contracts a whole copy of $\mathbb{P}^1$ to a point. For this reason, we call π a blow up. The inverse image of the origin is called the **exceptional divisor**.

Definition 6.1. *Let $\pi\colon X \longrightarrow Y$ be a birational morphism. The locus where π is not an isomorphism is called the* ***exceptional locus****. If $V \subset Y$, the inverse image of V is called the* ***total transform****. Let Z be the image of the exceptional locus. Suppose that V is not contained in Z. The* ***strict transform of*** *V is the closure of the inverse image of $V - Z$.*

It is interesting to compute the strict transform of some planar curves. We have already seen that lines through the origin lift to curves that sweep out the exceptional divisor. In fact the blow up separates the lines through the origin. These are then the fibres of the second morphism.

Let us now take a nodal cubic,

$$y^2 = x^2 + x^3.$$

We want to figure out its strict transform, so that we need the inverse image in the blow up. Outside the origin, there are two equations to be satisfied,

$$y^2 = x^2 + x^3 \qquad \text{and} \qquad xT = yS.$$

Passing to the coordinate patch $y = xt$, where $t = T/S$, and substituting for y in the first equation we get

$$x^2t^2 - x^2 - x^3 = x^2(t^2 - x - 1).$$

Now if $x = 0$, then $y = 0$, so that in fact locally $x = 0$ is the equation of the exceptional divisor. So the first factor just corresponds to the exceptional divisor. The second factor will tell us what the closure of our curve looks like, that is, the strict transform. Now over the origin, $x = 0$, so that $t^2 = 1$ and $t = \pm 1$. Thus our curve lifts to a curve which intersects the exceptional divisor in two points. (If we compute in the coordinate patch $x = sy$, we will see that the curve does not meet the point at infinity). These two points correspond to the fact that the nodal cubic has two tangent lines at the origin, one of slope 1 and one of slope -1.

Now consider what happens for the cuspidal cubic, $y^2 = x^3$. In this case we get

$$(xt)^2 - x^3 = x^2(t^2 - x).$$

Once again the factor of x^2 corresponds to the fact that the inverse image surely contains the exceptional divisor. But now we get the equation $t^2 = 0$, so that there is only one point over the origin, as one might expect from the geometry.

Let us go back to the Cremona transformation. To compute what gets blown up and blown down, it suffices to figure out what gets blown down, by symmetry. Consider the line $X = 0$. If $bc \neq 0$, the point $[0 : b : c]$ gets mapped to $[0 : 0 : 1]$. Thus the strict transform of the line $X = 0$ in the graph gets blown down to a point. By symmetry the strict transforms of the other two lines are also blown down to points. Outside of the union of these three lines, the map is clearly an isomorphism.

Thus the involution blows up the three points $[0 : 0 : 1]$, $[0 : 1 : 0]$, and $[1 : 0 : 0]$ and then blows down the three disjoint lines. Note that the three exceptional divisors become the three new coordinate lines.

One of the most impressive results of the nineteenth century is the following characterisation of the birational automorphism group of $\mathbb{P}^2$.

Theorem 6.2 (Noether)**.** *The birational automorphism group is generated by a Cremona transformation and* PGL(3)*.*

This result is very deceptive, since it is known that the birational automorphism group is, by any standards, very large.

One can blow up points on other varieties. For example, suppose we take $\mathbb{A}^3$, and consider the lines through the origin. This gives us a rational map to $\mathbb{P}^2$. The closure of the graph is the blow up of the origin; the exceptional divisor is a copy of $\mathbb{P}^2$. In coordinates we have (x, y, z) and $[S : T : U]$ and the equations for the graph are $xT = yS$, $xU = zS$ and $yU = zT$.

The blow up features in many interesting geometric constructions. Let $(XY - ZT = 0) \subset \mathbb{P}^3$ be the smooth quadric Q. Let $P_0 = [0 : 0 : 0 : 1]$, a point of the quadric. Consider projection from this point. We get a rational map to $\mathbb{P}^2$:

$$f : Q \dashrightarrow \mathbb{P}^2.$$

This map is in fact birational. A line meets the quadric in two points, one of which is P_0. f sends $[X : Y : Z : T]$ to $[X : Y : Z]$. The map

$$g : \mathbb{P}^2 \dashrightarrow Q,$$

which sends $[X : Y : Z]$ to $[X : Y : Z : XY/Z] = [XZ : YZ : Z^2 : XY]$ is the inverse. Let $\tilde{Q}$ be the strict transform of Q under the blow up. Locally about P_0, affine coordinates are (x, y, z), and the quadric is $z = xy$.

Let $[A : B : C]$ be coordinates on the exceptional divisor. Suppose we work on the affine patch $A \neq 0$. Then $y = bx$ and $z = cx$, where $b = B/A$ and $c = C/A$. Equations for the total transform of Q are

$$z - xy = cx - bx^2 = x(c - bx).$$

Equations for $\tilde{Q}$ are $c = bx$, smooth. When $b = 0$ we get $c = 0$, the equation of a line. $\tilde{Q} \longrightarrow Q$ contracts a copy E of $\mathbb{P}^1$ to a point.

Note that the induced morphism $h\colon Q \longrightarrow \mathbb{P}^2$ blows down the strict transform of the two lines passing through P_0 to two different points Q and R of $\mathbb{P}^2$. The image of the exceptional divisor E is the line connecting Q and R; with some patience one can check all of this in local coordinates.

We now consider how to define the blow up for an arbitrary scheme. Recall

(†) X is Noetherian, $\mathcal{S}_1$ is coherent, $\mathcal{S}$ is locally generated by $\mathcal{S}_1$.

Definition 6.3. *Let X be a Noetherian scheme and let $\mathcal{I}$ be a coherent sheaf of ideals on X. Let*

$$\mathcal{S} = \bigoplus_{d=0}^{\infty} \mathcal{I}^d,$$

where $\mathcal{I}^0 = \mathcal{O}_X$ and $\mathcal{I}^d$ is the dth power of $\mathcal{I}$. Then $\mathcal{S}$ satisfies (†).

*$\pi\colon \mathbf{Proj}\,\mathcal{S} \longrightarrow X$ is called the **blow up** of $\mathcal{I}$ (or Y, if Y is the subscheme of X associated to $\mathcal{I}$).*

Example 6.4. *Let $X = \mathbb{A}^n_k$ and let P be the origin. We check that we get the usual blow up. Let*

$$A = k[x_1, x_2, \dots, x_n].$$

As $X = \operatorname{Spec} A$ is affine and the ideal sheaf $\mathcal{I}$ of P is the sheaf associated to $\langle x_1, x_2, \dots, x_n \rangle$,

$$Y = \mathbf{Proj}\,\mathcal{S} = \operatorname{Proj} S,$$

where

$$S = \bigoplus_{d=0}^{\infty} I^d.$$

There is a surjective map

$$A[y_1, y_2, \dots, y_n] \longrightarrow S,$$

of graded rings, where y_i is sent to x_i. $Y \subset \mathbb{P}^n_A$ is the closed subscheme corresponding to this morphism. The kernel of this morphism is

$$\langle y_i x_j - y_j x_i \rangle,$$

which are the usual equations of the blow up.

Definition 6.5. *Let $f\colon X \longrightarrow Y$ be a morphism of schemes. We are going to define the **inverse image ideal sheaf** $\mathcal{I}' \subset \mathcal{O}_Y$. First we take the inverse image of the sheaf $f^{-1}\mathcal{I}$, where we just think of f as being a continuous map. Then $f^{-1}\mathcal{I} \subset f^{-1}\mathcal{O}_Y$. Let $\mathcal{I}' = f^{-1}\mathcal{I} \cdot \mathcal{O}_Y$ be the ideal generated by the image of $f^{-1}\mathcal{I}$ under the natural morphism $f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$.*

Theorem 6.6 (Universal Property of the blow up)**.** *Let X be a Noetherian scheme and let $\mathcal{I}$ be a coherent ideal sheaf.*

If $\pi\colon Y \longrightarrow X$ is the blow up of $\mathcal{I}$ then $\pi^{-1}\mathcal{I} \cdot \mathcal{O}_Y$ is an invertible sheaf. Moreover π is universal amongst all such morphisms. If $f\colon Z \longrightarrow X$ is any morphism such that $f^{-1}\mathcal{I} \cdot \mathcal{O}_Z$ is invertible then there is a unique induced morphism $g\colon Z \longrightarrow Y$ which makes the diagram commute

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[dr, "f"'] & Y \arrow[d, "\pi"] \\ & X. \end{tikzcd}$$

Proof. By uniqueness, we can check this locally. So we may assume that $X = \operatorname{Spec} A$ is affine. As $\mathcal{I}$ is coherent, it corresponds to an ideal $I \subset A$ and

$$X = \operatorname{Proj} \bigoplus_{d=0}^{\infty} I^d.$$

Now $\mathcal{O}_Y(1)$ is an invertible sheaf on Y. It is not hard to check that $\pi^{-1}\mathcal{I} \cdot \mathcal{O}_Y = \mathcal{O}_Y(1)$.

Now show that we are given $f\colon Z \longrightarrow X$. We first construct g, then show that if g is any factorisation of f, the pullback ideal sheaf is invertible and then finally show that g is unique.

Pick generators $a_0, a_1, \dots, a_n$ for I. This gives rise to a surjective map of graded A-algebras

$$\phi\colon A[x_0, x_1, \dots, x_n] \longrightarrow \bigoplus_{d=0}^{\infty} I^d,$$

whence to a closed immersion $Y \subset \mathbb{P}^n_A$. The kernel of ϕ is generated by all homogeneous polynomials $F(x_0, x_1, \dots, x_n)$ such that $F(a_0, a_1, \dots, a_n) = 0$.

Now the elements $a_0, a_1, \dots, a_n$ pullback to global sections $s_0, s_1, \dots, s_n$ of the invertible sheaf $\mathcal{L} = f^{-1}\mathcal{I} \cdot \mathcal{O}_Y$ and $s_0, s_1, \dots, s_n$ generate $\mathcal{L}$. So we get a morphism

$$g\colon Z \longrightarrow \mathbb{P}^n_A,$$

over X, such that $g^*\mathcal{O}_{\mathbb{P}^n_A}(1) = \mathcal{L}$ and $g^{-1}x_i = s_i$. Suppose that $F(x_0, x_1, \dots, x_n)$ is a homogeneous polynomial in the kernel of ϕ. Then $F(a_0, a_1, \dots, a_n) = 0$ so that $F(s_0, s_1, \dots, s_n) = 0$ in $H^0(Z, \mathcal{L}^d)$. It follows that g factors through Y.

Now suppose that $f\colon Z \longrightarrow X$ factors through $g\colon Z \longrightarrow Y$. Then

$$f^{-1}\mathcal{I} \cdot \mathcal{O}_Z = g^{-1}(\mathcal{I} \cdot \mathcal{O}_Y) \cdot \mathcal{O}_Z = g^{-1}\mathcal{O}_Y(1) \cdot \mathcal{O}_Z.$$

Thus $\mathcal{L} = f^{-1}\mathcal{I} \cdot \mathcal{O}_Z$ is an invertible sheaf.

Therefore there is a surjective map

$$g^*\mathcal{O}_Y(1) \longrightarrow \mathcal{L}.$$

But then this map must be an isomorphism and so $g^*\mathcal{O}_Y(1) = \mathcal{L}$. $s_i = g^* x_i$ and uniqueness follows. $\square$

Note that by the universal property, the morphism π is an isomorphism outside of the subscheme V defined by $\mathcal{I}$. We may put the universal property differently. The only subscheme with an invertible ideal sheaf is a Cartier divisor (local generators of the ideal, give local equations for the Cartier divisor). So the blow up is the smallest morphism which turns a subscheme into a Cartier divisor. Perhaps surprisingly, therefore, blowing up a Weil divisor might give a non-trivial birational map.

If X is a variety it is not hard to see that π is a projective, birational morphism. In particular if X is quasi-projective or projective then so is Y. We note that there is a converse to this:

Theorem 6.7. *Let X be a quasi-projective variety and let $f\colon Z \longrightarrow X$ be a birational projective morphism.*

Then there is a coherent ideal sheaf $\mathcal{I}$ and a commutative diagram

$$\begin{tikzcd} Z \arrow[rr] \arrow[dr, "f"'] & & Y \arrow[dl, "\pi"] \\ & X, & \end{tikzcd}$$

where $\pi\colon Y \longrightarrow X$ is the blow up of $\mathcal{I}$ and the top row is an isomorphism.