5. Relative proj and projective bundles

We want to define a relative version of Proj, in pretty much the same way we defined a relative version of Spec. We start with a scheme X and a quasi-coherent sheaf S sheaf of graded \mathcal{O}_X -algebras,

$$\mathcal{S} = \bigoplus_{d \in \mathbb{N}} \mathcal{S}_d$$

where $S_0 = \mathcal{O}_X$. It is convenient to make some simplifying assumptions:

(†) X is Noetherian, S_1 is coherent, S is locally generated by S_1 .

To construct relative Proj, we cover X by open affines $U = \operatorname{Spec} A$. With a view towards what comes next, we denote global sections of Sover U by $H^0(U, S)$. Then $S(U) = H^0(U, S)$ is a graded A-algebra, and we get π_U : Proj $S(U) \longrightarrow U$ a projective morphism. If $f \in A$ then we get a commutative diagram

$$\begin{array}{c|c} \operatorname{Proj} \mathcal{S}(U_f) \longrightarrow \operatorname{Proj} \mathcal{S}(U) \\ \pi_{U_f} & & \pi_U \\ U_f & & & U. \end{array}$$

It is not hard to glue π_U together to get π : **Proj** $\mathcal{S} \longrightarrow X$. We can also glue the invertible sheaves together to get an invertible sheaf $\mathcal{O}(1)$.

The relative construction has some similarities to the old construction.

Example 5.1. If X is Noetherian and

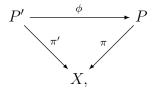
$$\mathcal{S} = \mathcal{O}_X[T_0, T_1, \dots, T_n],$$

then satisfies (†) and $\operatorname{Proj} S = \mathbb{P}_X^n$.

Given a sheaf S satisfying (†), and an invertible sheaf \mathcal{L} , it is easy to construct a quasi-coherent sheaf $S' = S \star \mathcal{L}$, which satisfies (†). The graded pieces of S' are $S_d \otimes \mathcal{L}^d$ and the multiplication maps are the obvious ones. There is a natural isomorphism

$$\phi \colon P' = \operatorname{\mathbf{Proj}} \mathcal{S}' \longrightarrow P = \operatorname{\mathbf{Proj}} \mathcal{S},$$

which makes the diagram commute



and

$$\phi^* \mathcal{O}_P(1) \simeq \mathcal{O}_{P'}(1) \otimes \pi'^* \mathcal{L}$$

Note that π is always proper; in fact π is projective over any open affine and properness is local on the base. Even better π is projective if X has an ample line bundle; see (II.7.10).

There are two very interesting family of examples of the construction of relative Proj. Suppose that we start with a locally free sheaf \mathcal{E} of rank $r \geq 2$. Note that

$$\mathcal{S} = \bigoplus \operatorname{Sym}^d \mathcal{E},$$

satisfies (†). $\mathbb{P}(\mathcal{E}) = \operatorname{Proj} \mathcal{S}$ is the projective bundle over X associated to \mathcal{E} . The fibres of $\pi \colon \mathbb{P}(\mathcal{E}) \longrightarrow X$ are copies of \mathbb{P}^n , where n = r - 1. We have

$$\bigoplus_{l=0}^{\infty} \pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(l) = \mathcal{S},$$

so that in particular

$$\pi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = \mathcal{E}.$$

Also there is a natural surjection

$$\pi^* \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1).$$

Indeed, it suffices to check both statements locally, so that we may assume that X is affine. The first statement is standard and proved in 18.725, and the second statement reduces to the statement that the sections x_0, x_1, \ldots, x_n generate $\mathcal{O}_P(1)$. The most interesting result is:

Proposition 5.2. Let $g: Y \longrightarrow X$ be a morphism.

Then a morphism $f: Y \longrightarrow \mathbb{P}(\mathcal{E})$ over X is the same as giving an invertible sheaf \mathcal{L} on Y and a surjection $g^*\mathcal{E} \longrightarrow \mathcal{L}$.

Proof. One direction is clear; if $f: Y \longrightarrow \mathbb{P}(\mathcal{E})$ is a morphism over X, then the surjective morphism of sheaves

$$\pi^* \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1),$$

pullsback to a surjective morphism

$$g^*\mathcal{E} = f^*(\pi^*\mathcal{E}) \longrightarrow \mathcal{L} = f^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1).$$

Conversely suppose we are given an invertible sheaf \mathcal{L} and a surjective morphism of sheaves

$$g^*\mathcal{E}\longrightarrow \mathcal{L}.$$

I claim that there is then a unique morphism $f: Y \longrightarrow \mathbb{P}(\mathcal{E})$ over X, which induces the given surjection. By uniqueness, it suffices to prove this result locally. So we may assume that $X = \operatorname{Spec} A$ is affine and

$$\mathcal{E} = \bigoplus_{\substack{i=0\\2}}^{n} \mathcal{O}_X,$$

is free. In this case surjectivity reduces to the statement that the images s_0, s_1, \ldots, s_n of the standard sections generate \mathcal{L} , and the result reduces to one we have already proved.