4. Ample line bundles on toric varieties

It is interesting to find the ample line bundles on a toric variety. Suppose that $X = X(F)$ is the toric variety associated to the fan $F \subset N_{\mathbb{R}}$. Recall that we can associate to a T-Cartier divisor $D = \sum a_i D_i$, a continuous piecewise linear function

$$\phi_D \colon |F| \longrightarrow \mathbb{R},$$

where $|F| \subset N_{\mathbb{R}}$ is the support of the fan, which is specified by the rule that $\phi_D(v_i) = -a_i$.

We can also associate to D a rational polyhedron

$$\begin{aligned} P_D &= \{\, u \in M_{\mathbb{R}} \,|\, \langle u, v_i \rangle \geq -a_i \quad \forall i \,\} \\ &= \{\, u \in M_{\mathbb{R}} \,|\, u \geq \phi_D \,\}. \end{aligned}$$

Lemma 4.1. *If X is a toric variety and D is T-Cartier then*

$$H^0(X, \mathcal{O}_X(D)) = \bigoplus_{u \in P_D \cap M} k \cdot \chi^u.$$

Proof. Suppose that $\sigma \in F$ is a cone. We can identify $H^0(U_\sigma, \mathcal{O}_{U_\sigma}(D))$ as the set of rational functions f on X which have a pole no worse than D:

$$(f) + D \geq 0.$$

This gives us a vector space of rational functions which, as usual, decomposes into eigenspaces. Now f has no poles along the torus, so we may assume that f belongs to the Laurent polynomial ring

$$K[x_1, x_1^{-1}, x_2, x_2^{-1}, x_3, x_3^{-1}, \dots, x_n, x_n^{-1}].$$

Therefore the eigenspaces are given by χ^u, $u \in M$ and we want

$$(\chi^u) + D \geq 0.$$

Writing this out in components, we have

$$\langle u, v_i \rangle + a_i \geq 0 \qquad \text{for all} \qquad v_i \in \sigma.$$

Thus we have

$$H^0(U_\sigma, \mathcal{O}_{U_\sigma}(D)) = \bigoplus_{u \in P_D(\sigma) \cap M} k \cdot \chi^u,$$

where

$$P_D(\sigma) = \{\, u \in M_{\mathbb{R}} \,|\, \langle u, v_i \rangle \geq -a_i \quad \forall v_i \in \sigma \,\}.$$

These identifications are compatible on overlaps. Since

$$H^0(X, \mathcal{O}_X(D)) = \bigcap_{\sigma \in F} H^0(U_\sigma, \mathcal{O}_{U_\sigma}(D))$$

and

$$P_D = \bigcap_{\sigma \in F} P_D(\sigma),$$

the result is clear. □

It is interesting to compute some examples. Let's start with $\mathbb{P}^1$. A T-Cartier divisor is a sum $ap + bq$ (p and q are the fixed points, zero and infinity). We want those rational functions which have a pole no worse than $-a$ at p and a pole no worse than $-b$ at q. Consider the general monomial $f = x^i$. If $i \geq 0$ then f is regular at p and has a pole of order i at q. So $i \leq a$. If $i \leq 0$ then f has a pole of order $-i$ at p and f is regular at q. So $-i \leq b$, that is $i \geq -b$.

The polytope corresponding to $ap+bq$ is $[-b, a]$ and a general rational function with poles no worse than $ap + bq$ has the form

$$c_{-b}x^{-b} + c_{-b+1}x^{-b+1} + \cdots + c_{-1}x^{-1} + c_0 + c_1 x + \cdots + c_a x^a.$$

The corresponding piecewise linear function is

$$\phi(x) = \begin{cases} -ax & x > 0 \\ bx & x < 0. \end{cases}$$

Now consider $\mathbb{P}^2$ and dD_3. We are looking at rational functions $x^i y^j$ which are regular on the standard open affine $U_0 = \mathbb{A}^2_K$. So $i \geq 0$ and $j \geq 0$. Since we have a pole no worse than d along D_3, we must have $i + j \leq d$. Therefore P_D is the convex hull of $(0, 0)$, $(d, 0)$ and $(0, d)$. The number of integral points is

$$\frac{(d+1)^2}{2} + \frac{d+1}{2} = \frac{(d+2)(d+1)}{2},$$

which is the usual formula for the number of homogeneous polynomials of degree d in three variables.

Let D be a Cartier divisor on a toric variety $X = X(F)$ given by a fan F. It is interesting to consider when the complete linear system $|D|$ is base point free. Since any Cartier divisor is linearly equivalent to a T-Cartier divisor, we might as well suppose that $D = \sum a_i D_i$ is T-Cartier. Note that the base locus of the complete linear system of any Cartier divisor is invariant under the action of the torus. Since there are only finitely many orbits, it suffices to show that for each cone $\sigma \in F$ the point $x_\sigma \in U_\sigma$ is not in the base locus. It is also clear that if x_σ is not in the base locus of $|D|$ then in fact one can find a T-Cartier divisor $D' \in |D|$ which does not contain x_σ.

Now the invariant Weil divisor D_i contains x_σ precisely when $v_i \in \sigma$. So we want an invariant Weil divisor $D' = \sum b_i D_i$ such that $b_i \geq 0$

with strict equality if $v_i \notin \sigma$. As $D' = D + (\chi^u)$, if x_σ is not in the base locus of $|D|$ then we can find $u \in M$ such that

$$\langle u, v_i \rangle \geq -a_i,$$

with strict equality if $v_i \in \sigma$. The interesting thing is that we can reinterpret this condition using ϕ_D.

Definition 4.2. *The function $\phi \colon V \longrightarrow \mathbb{R}$ is (upper)* ***convex*** *if*

$$\phi(\lambda v + (1-\lambda)w) \geq \lambda\phi(v) + (1-\lambda)w \qquad \forall v, w \in V.$$

When we have a fan F and ϕ is linear on each cone σ, then ϕ is called ***strictly convex*** *if the linear functions $u(\sigma)$ and $u(\sigma')$ are different, for different maximal cones σ and σ'.*

Theorem 4.3. *Let $X = X(F)$ be the toric variety associated to a T-Cartier divisor D.*

Then

(1) $|D|$ is base point free if and only if ψ_D is convex.
(2) D is very ample if and only if ψ_D is strictly convex and the semigroup S_σ is generated by

$$\{\, u - u(\sigma) \,|\, u \in P_D \cap M \,\}.$$

Proof. (1) follows from the remarks above. (2) is proved in Fulton's book. □

For example if $X = \mathbb{P}^1$ and

$$\phi(x) = \begin{cases} -ax & x > 0 \\ bx & x < 0. \end{cases}$$

so that $D = ap + bq$ then ϕ is convex if and only if $a + b \geq 0$ in which case D is base point free. D is very ample if and only if $a + b > 0$. When ϕ is continuous and linear on each cone σ, we may restate the convex condition as saying that the graph of ϕ lies under the graph of $u(\sigma)$. It is strictly convex if it lies strictly under the graph of $u(\sigma)$ outside of σ, for all n-dimensional cones σ.

Using this, it is altogether too easy to give an example of a smooth proper toric variety which is not projective. Let $F \subset N_{\mathbb{R}} = \mathbb{R}^3$ given by the edges $v_1 = -e_1$, $v_2 = -e_2$, $v_3 = -e_3$, $v_4 = e_1 + e_2 + e_3$, $v_5 = v_3 + v_4$, $v_6 = v_1 + v_4$ and $v_7 = v_2 + v_4$. Now connect v_1 to v_5, v_3 to v_7 and v_2 to v_6 and v_5 to v_6, v_6 to v_7 and v_7 to v_5.

It is not hard to check that X is smooth and proper (proper translates to the statement that the support $|F|$ of the fan is the whole of $N_{\mathbb{R}}$).

Suppose that ψ is strictly convex. Let

$$w = \frac{v_1 + v_3 + v_4}{3},$$

the barycentric centre of the triangle with vertices v_1, v_3 and v_4. Then

$$w = \frac{v_1 + v_5}{3} = \frac{v_3 + v_6}{3}.$$

Since v_1 and v_5 belong to the same maximal cone, ψ is linear on the line connecting them. In particular

$$\psi(w) = \psi(\frac{v_1 + v_5}{3}) = \frac{1}{3}\psi(v_1) + \frac{1}{3}\psi(v_5).$$

Since v_1, v_5 and v_3 belong to the same cone and v_6 does not, by strict convexity,

$$\psi(w) = \psi(\frac{v_3 + v_6}{3}) > \frac{1}{3}\psi(v_3) + \frac{1}{3}\psi(v_6).$$

Putting all of this together, we get

$$\psi(v_1) + \psi(v_5) > \psi(v_2) + \psi(v_6).$$

By symmetry

$$\begin{aligned}
\psi(v_1) + \psi(v_5) &> \psi(v_3) + \psi(v_6) \\
\psi(v_2) + \psi(v_6) &> \psi(v_1) + \psi(v_7) \\
\psi(v_3) + \psi(v_7) &> \psi(v_2) + \psi(v_5).
\end{aligned}$$

But adding up these three inequalities gives a contradiction.