3. Divisors on toric varieties

We start with computing the class group of a toric variety. Recall that the class group is the group of Weil divisors modulo linear equivalence. We denote the class group either by $\text{Cl}(X)$ or $A_{n-1}(X)$.

When talking about Weil divisors, we will always assume we have a scheme which is:

(*) noetherian, integral, separated, and regular in codimension one.

This is never a problem for toric varieties. If $X$ is a toric variety, by assumption there is a dense open subset $U \cong \mathbb{G}_m^n$. The complement $Z$ is a closed invariant subset.

Lemma 3.1. Suppose that $X$ satisfies (*), let $Z$ be a closed subset and let $U = X \setminus Z$.

Then there is an exact sequence

$$Z^s \rightarrow \text{Cl}(X) \rightarrow \text{Cl}(U) \rightarrow 0,$$

where $s$ is the number of components of $Z$ which are prime divisors.

Proof. If $Y$ is a prime divisor on $X$ then $Y' = Y \cap U$ is either a prime divisor on $U$ or empty. This defines a group homomorphism $\rho: \text{Div}(X) \rightarrow \text{Div}(U)$.

If $Y' \subset U$ is a prime divisor, then let $Y$ be the closure of $Y'$ in $X$. Then $Y$ is a prime divisor and $Y' = Y \cap U$. Thus $\rho$ is surjective. If $f$ is a rational function on $X$ and $Y = (f)$, then the image of $Y$ in $\text{Div}(U)$ is equal to $(f|_U)$, so $\rho$ descends to a map of class groups.

If $Z = Z' \cup \bigcup_{i=1}^{s} Z_i$ where $Z'$ has codimension at least two and $Z_1, Z_2, \ldots, Z_s$ is a prime divisor, then the map which sends $(m_1, m_2, \ldots, m_s)$ to $\sum m_i Z_i$ generates the kernel. \qed

Example 3.2. Let $X = \mathbb{P}^2_K$ and $C$ be an irreducible curve of degree $d$. Then $\text{Cl}(\mathbb{P}^2 - C)$ is equal to $\mathbb{Z}_d$. Similarly $\text{Cl}(\mathbb{A}^n_K) = 0$.

Back to assuming that $X$ is a toric variety. It follows by (3.1) that there is an exact sequence

$$Z^s \rightarrow \text{Cl}(X) \rightarrow \text{Cl}(U) \rightarrow 0.$$

Applying this to $X = \mathbb{A}^n_K$ it follows that $\text{Cl}(U) = 0$. So we get an exact sequence

$$0 \rightarrow K \rightarrow Z^s \rightarrow \text{Cl}(X) \rightarrow 0.$$

We want to identify the kernel $K$. This is equal to the set of principal divisors which are supported on the invariant divisors. If $f$ is a rational function such that $(f)$ is supported on the invariant divisors then $f$ has
no zeroes or poles on the torus; it follows that $f = \lambda \chi^u$, where $\lambda \in K^*$ and $u \in M$.

Hence there is an exact sequence

$$M \longrightarrow \mathbb{Z}^* \longrightarrow \text{Cl}(X) \longrightarrow 0.$$ 

Recall that the invariant divisors are in bijection with the one dimensional cones $\tau$ of the fan $F$. Now, given a one dimensional cone $\tau$, there is a unique vector $v \in \tau \cap M$ such that if $w$ also belongs to $\tau \cap M$ and we write $w = p \cdot v$ then $p \geq 1$. We call $v$ a primitive generator of $\tau$.

**Lemma 3.3.** Let $u \in M$. Suppose that $X$ is the affine toric variety associated to a cone $\sigma \subset N_\mathbb{R}$. Let $v$ be a primitive generator of a one dimensional ray $\tau$ of $\sigma$ and let $D$ be the corresponding invariant divisor. Then $\text{ord}_D(\chi^u) = \langle u, v \rangle$. In particular

$$\langle \chi^u \rangle = \sum \langle u, v_i \rangle D_i,$$

where the sum ranges over the invariant divisors.

**Proof.** We can calculate the order on the open set $U_\sigma = \mathbb{A}^1_k \times \mathbb{G}_m^{n-1}$, where $D$ corresponds to $\{0\} \times \mathbb{G}_m^{n-1}$. In this case we can ignore the factor $\mathbb{G}_m^{n-1}$ and we are reduced to the one dimensional case. So $N = \mathbb{Z}$, $v = 1$ and $u \in M = \mathbb{Z}$. In this case $\chi^u$ is the monomial $x^u$ and the order of vanishing at the origin is exactly $u$. \hfill \Box

It follows that if $X = X(F)$ is the toric variety associated to a fan $F$ which spans $N_\mathbb{R}$ then we have a short exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^* \longrightarrow \text{Cl}(X) \longrightarrow 0.$$

**Example 3.4.** Let $\sigma$ be the cone spanned by $2e_1 - e_2$ and $e_2$ inside $N_\mathbb{R} = \mathbb{R}^2$. There are two invariant divisors $D_1$ and $D_2$. The principal divisor associated to $u = f_1 = (1, 0)$ is $2D_1$ and the principal divisor associated to $u = f_2 = (0, 1)$ is $D_2 - D_1$. So the class group is $\mathbb{Z}_2$.

Note that the dual cone $\bar{\sigma}$ is the cone spanned by $f_1$ and $f_1 + 2f_2$. Generators for the monoid $S_\sigma = \bar{\sigma} \cap M$ are $f_1, f_1 + f_2$ and $f_1 + 2f_2$. So the group algebra

$$A_\sigma = k[x, xy, xy^2] = \frac{k[u, v, w]}{\langle u^2 - uw \rangle},$$

and $X = U_\sigma$ is the quadric cone.

Now suppose we take the standard fan associated to $\mathbb{P}^2$. The invariant divisors are the three coordinate lines, $D_1, D_2$ and $D_3$. If $f_1 = (1, 0)$
and \( f_2 = (0, 1) \) then
\[
(\chi^{f_1}) = D_1 - D_3 \quad \text{and} \quad (\chi^{f_2}) = D_2 - D_3.
\]

So the class group is \( \mathbb{Z} \).

We now turn to calculating the Picard group of a toric variety \( X \).

**Definition 3.5.** Let \( X \) be a scheme.

The set of invertible sheaves forms an abelian group \( \text{Pic} (X) \), where multiplication corresponds to tensor product and the inverse to the dual.

Recall that if \( X \) is a normal variety, every Cartier divisor \( D \) on \( X \) determines a Weil divisor
\[
\text{ord}_V(D)V,
\]
where sum runs over all prime divisors of \( X \). Thus the set of Cartier divisors embeds in the set of Weil divisors. We say that \( X \) is **factorial** if every Weil divisor is Cartier.

Let’s consider which Weil divisors on a toric variety are Cartier. We classify all Cartier divisors whose underlying Weil divisor is invariant; we dub these Cartier divisors \( T \)-Cartier. We start with the case of the affine toric variety associated to a cone \( \sigma \subset \mathbb{N}_R \). It suffices to classify all invertible subsheaves \( \mathcal{O}_X(D) \subset \mathcal{K} \), where \( \mathcal{K} \) is the sheaf of total quotient rings of \( \mathcal{O}_X \). Taking global sections, since we are on an affine variety, it suffices to classify all fractional ideals,
\[
I = H^0(X, \mathcal{O}_X(D)) \subset H^0(X, \mathcal{K}).
\]

Invariance of \( D \) implies that \( I \) is graded by \( M \), that is, \( I \) is a direct sum of eigenspaces. As \( D \) is Cartier, \( I \) is principal at the distinguished point \( x_\sigma \) of \( U_\sigma \), so that \( I/mI \) is one dimensional, where
\[
m = \sum k \cdot \chi^u.
\]

Pick \( U \in M \) such that the image of \( \chi^u \) generates this one dimensional vector space. Nakayama’s Lemma implies that \( I = A_\sigma \chi^u \), that is \( I \) is the ideal generated by \( \chi^u \), so that \( D = (\chi^u) \) is principal. As every Weil divisor is linearly equivalent to a Weil divisor supported on the invariant divisors, every Cartier divisor is linearly equivalent to a \( T \)-Cartier divisor. Hence, the only Cartier divisors are the principal divisors and \( X \) is factorial if and only if the Class group is trivial.

**Example 3.6.** The quadric cone \( Q \), given by \( xy - z^2 = 0 \) in \( \mathbb{A}^3_k \) is not factorial. We have already seen (3.4) that the class group is \( \mathbb{Z}_2 \).
If $\sigma \subset N_\mathbb{R}$ is not maximal dimensional then every Cartier divisor on $U_\sigma$ whose associated Weil divisor is invariant is of the form $(\chi^u)$ but

$$(\chi^u) = (\chi^{u'}) \quad \text{if and only if} \quad u - u' \in \sigma^\perp \cap M = M(\sigma).$$

So the $T$-Cartier divisors are in correspondence with $M/M(\sigma)$.

Now suppose that $X = X(F)$ is a general toric variety. Then a $T$-Cartier divisor is given by specifying an element $u(\sigma) \in M/M(\sigma)$, for every cone $\sigma$ in $F$. This defines a divisor $(\chi^{-u(\sigma)})$; equivalently a fractional ideal

$$I = H^0(X, \mathcal{O}_X(D)) = A_\sigma \cdot \chi^{u(\sigma)}.$$

These maps must agree on overlaps; if $\tau$ is a face of $\sigma$ then $u(\sigma) \in M/M(\sigma)$ must map to $u(\tau) \in M/M(\tau)$.

Note that it is somewhat hard to keep track of the $T$-Cartier divisors. We look for a way to repackage the same combinatorial data into a more convenient form. As usual, this means we should look at the dual picture.

The data

$$\{ u(\sigma) \in M/M(\sigma) \mid \sigma \in F \},$$

for a $T$-Cartier divisor $D$ determines a continuous piecewise linear function $\phi_D$ on the support $|F|$ of $F$. If $v \in \sigma$ then let

$$\phi_D(v) = \langle u(\sigma), v \rangle.$$  

Compatibility of the data implies that $\phi_D$ is well-defined and continuous. Conversely, given any continuous function $\phi$, which is linear and integral (that is, given by an element of $M$) on each cone, we can associate a unique $T$-Cartier divisor $D$. If $D = \sum a_i D_i$ the function is given by $\phi_D(v_i) = -a_i$, where $v_i$ is the primitive generator of the ray corresponding to $D_i$.

Note that

$$\phi_D + \phi_E = \phi_{D+E} \quad \text{and} \quad \phi_{mD} = m\phi_D.$$  

Note also that $\phi_{(\chi^u)}$ is the linear function given by $-u$. So $D$ and $E$ are linearly equivalent if and only if $\phi_D$ and $\phi_E$ differ by a linear function.

If $X$ is any variety which satisfies (\footnote{[1]} then the natural map

$$\text{Pic}(X) \rightarrow \text{Cl}(X),$$

is an embedding. It is an interesting to compare Pic$(X)$ and Cl$(X)$ on a toric variety. Denote by Div$_T(X)$ the group of $T$-Cartier divisors.
Proposition 3.7. Let $X = X(F)$ be the toric variety associated to a fan $F$ which spans $N_{\mathbb{R}}$. Then there is a commutative diagram with exact rows:

$$
\begin{array}{c}
0 \rightarrow M \rightarrow \text{Div}_T(X) \rightarrow \text{Pic}(X) \rightarrow 0 \\
\| \downarrow \quad \downarrow \quad \downarrow \\
0 \rightarrow M \rightarrow \mathbb{Z}^s \rightarrow \text{Cl}(X) \rightarrow 0.
\end{array}
$$

In particular

$$\rho(X) = \text{rank}(\text{Pic}(X)) \leq \text{rank}(\text{Cl}(X)) = s - n.$$

Further $\text{Pic}(X)$ is a free abelian group.

Proof. We have already seen that the bottom row is exact. If $L$ is an invertible sheaf then $L|_U$ is trivial. Suppose that $L = \mathcal{O}_X(E)$. Pick a rational function such that $(f)|_U = E|_U$. Let $D = E - (f)$. Then $D$ is $T$-Cartier, since it is supported away from the torus and exactness of the top row is easy.

Finally, $\text{Pic}(X)$ is represented by equivalence classes of continuous, piecewise integral linear functions modulo linear functions. Clearly if $m\phi$ is linear then so is $\phi$, so that $\text{Pic}(X)$ is torsion free. \qed