3. Divisors on toric varieties

We start with computing the class group of a toric variety. Recall that the class group is the group of Weil divisors modulo linear equivalence. We denote the class group either by $\mathrm{Cl}(X)$ or $A_{n-1}(X)$.

When talking about Weil divisors, we will always assume we have a scheme which is:

$(*)$ noetherian, integral, separated, and regular in codimension one.

This is never a problem for toric varities. If X is a toric variety, by assumption there is a dense open subset $U \simeq \mathbb{G}_m^n$. The complement Z is a closed invariant subset.

Lemma 3.1. *Suppose that X satisfies $(*)$, let Z be a closed subset and let $U = X \setminus Z$.*

Then there is an exact sequence

$$\mathbb{Z}^s \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U) \longrightarrow 0,$$

where s is the number of components of Z which are prime divisors.

Proof. If Y is a prime divisor on X then $Y' = Y \cap U$ is either a prime divisor on U or empty. This defines a group homomorphism

$$\rho\colon \mathrm{Div}(X) \longrightarrow \mathrm{Div}(U).$$

If $Y' \subset U$ is a prime divisor, then let Y be the closure of Y' in X. Then Y is a prime divisor and $Y' = Y \cap U$. Thus ρ is surjective. If f is a rational function on X and $Y = (f)$, then the image of Y in $\mathrm{Div}(U)$ is equal to $(f|_U)$, so ρ descends to a map of class groups.

If $Z = Z' \cup \bigcup_{i=1}^s Z_i$ where Z' has codimension at least two and $Z_1, Z_2, \dots, Z_s$ is a prime divisor, then the map which sends $(m_1, m_2, \dots, m_s)$ to $\sum m_i Z_i$ generates the kernel. □

Example 3.2. *Let $X = \mathbb{P}^2_K$ and C be an irreducible curve of degree d. Then $\mathrm{Cl}(\mathbb{P}^2 - C)$ is equal to $\mathbb{Z}_d$. Similarly $\mathrm{Cl}(\mathbb{A}^n_K) = 0$.*

Back to assuming that X is a toric variety. It follows by (3.1) that there is an exact sequence

$$\mathbb{Z}^s \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U) \longrightarrow 0.$$

Applying this to $X = \mathbb{A}^n_K$ it follows that $\mathrm{Cl}(U) = 0$. So we get an exact sequence

$$0 \longrightarrow K \longrightarrow \mathbb{Z}^s \longrightarrow \mathrm{Cl}(X) \longrightarrow 0.$$

We want to identify the kernel K. This is equal to the set of principal divisors which are supported on the invariant divisors. If f is a rational function such that (f) is supported on the invariant divisors then f has no zeroes or poles on the torus; it follows that $f = \lambda\chi^u$, where $\lambda \in K^*$ and $u \in M$.

Hence there is an exact sequence

$$M \longrightarrow \mathbb{Z}^s \longrightarrow \mathrm{Cl}(X) \longrightarrow 0.$$

Recall that the invariant divisors are in bijection with the one dimensional cones τ of the fan F. Now, given a one dimensional cone τ, there is a unique vector $v \in \tau \cap M$ such that if w also belongs to $\tau \cap M$ and we write $w = p \cdot v$ then $p \geq 1$. We call v a **primitive generator** of τ.

Lemma 3.3. *Let $u \in M$. Suppose that X is the affine toric variety associated to a cone $\sigma \subset N_{\mathbb{R}}$. Let v be a primitive generator of a one dimensional ray τ of σ and let D be the corresponding invariant divisor.*

Then $\mathrm{ord}_D(\chi^u) = \langle u, v\rangle$. In particular

$$(\chi^u) = \sum_i \langle u, v_i\rangle D_i,$$

where the sum ranges over the invariant divisors.

Proof. We can calculate the order on the open set $U_\tau = \mathbb{A}^1_k \times \mathbb{G}_m^{n-1}$, where D corresponds to $\{0\} \times \mathbb{G}_m^{n-1}$. In this case we can ignore the factor $\mathbb{G}_m^{n-1}$ and we are reduced to the one dimensional case. So $N = \mathbb{Z}$, $v = 1$ and $u \in M = \mathbb{Z}$. In this case χ^u is the monomial x^u and the order of vanishing at the origin is exactly u. $\square$

It follows that if $X = X(F)$ is the toric variety associated to a fan F which spans $N_{\mathbb{R}}$ then we have a short exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^s \longrightarrow \mathrm{Cl}(X) \longrightarrow 0.$$

Example 3.4. *Let σ be the cone spanned by $2e_1 - e_2$ and e_2 inside $N_{\mathbb{R}} = \mathbb{R}^2$. There are two invariant divisors D_1 and D_2. The principal divisor associated to $u = f_1 = (1, 0)$ is $2D_1$ and the principal divisor associated to $u = f_2 = (0, 1)$ is $D_2 - D_1$. So the class group is $\mathbb{Z}_2$.*

Note that the dual cone $\check{\sigma}$ is the cone spanned by f_1 and $f_1 + 2f_2$. Generators for the monoid $S_\sigma = \check{\sigma} \cap M$ are f_1, $f_1 + f_2$ and $f_1 + 2f_2$. So the group algebra

$$A_\sigma = k[x, xy, xy^2] = \frac{k[u, v, w]}{\langle v^2 - uw\rangle},$$

and $X = U_\sigma$ is the quadric cone.

Now suppose we take the standard fan associated to $\mathbb{P}^2$. The invariant divisors are the three coordinate lines, D_1, D_2 and D_3. If $f_1 = (1, 0)$

and $f_2 = (0,1)$ *then*

$$(\chi^{f_1}) = D_1 - D_3 \qquad \textit{and} \qquad (\chi^{f_2}) = D_2 - D_3.$$

So the class group is $\mathbb{Z}$.

We now turn to calculating the Picard group of a toric variety X.

Definition 3.5. *Let* X *be a scheme.*

The set of invertible sheaves forms an abelian group $\operatorname{Pic}(X)$*, where multiplication corresponds to tensor product and the inverse to the dual.*

Recall that if X is a normal variety, every Cartier divisor D on X determines a Weil divisor

$$\operatorname{ord}_V(D)V,$$

where sum runs over all prime divisors of X. Thus the set of Cartier divisors embeds in the set of Weil divisors. We say that X is **factorial** if every Weil divisor is Cartier.

Let's consider which Weil divisors on a toric variety are Cartier. We classify all Cartier divisors whose underlying Weil divisor is invariant; we dub these Cartier divisors T-Cartier. We start with the case of the affine toric variety associated to a cone $\sigma \subset N_{\mathbb{R}}$. It suffices to classify all invertible subsheaves $\mathcal{O}_X(D) \subset \mathcal{K}$, where $\mathcal{K}$ is the sheaf of total quotient rings of $\mathcal{O}_X$. Taking global sections, since we are on an affine variety, it suffices to classify all fractional ideals,

$$I = H^0(X, \mathcal{O}_X(D)) \subset H^0(X, \mathcal{K}).$$

Invariance of D implies that I is graded by M, that is, I is a direct sum of eigenspaces. As D is Cartier, I is principal at the distinguished point x_σ of U_σ, so that $I/\mathfrak{m}I$ is one dimensional, where

$$\mathfrak{m} = \sum k \cdot \chi^u.$$

Pick $U \in M$ such that the image of χ^u generates this one dimensional vector space. Nakayama's Lemma implies that $I = A_\sigma \chi^u$, that is I is the ideal generated by χ^u, so that $D = (\chi^u)$ is principal. As every Weil divisor is linearly equivalent to a Weil divisor supported on the invariant divisors, every Cartier divisor is linearly equivalent to a T-Cartier divisor. Hence, the only Cartier divisors are the principal divisors and X is factorial if and only if the Class group is trivial.

Example 3.6. *The quadric cone* Q*, given by* $xy - z^2 = 0$ *in* $\mathbb{A}^3_k$ *is not factorial. We have already seen* (3.4) *that the class group is* $\mathbb{Z}_2$.

If $\sigma \subset N_{\mathbb{R}}$ is not maximal dimensional then every Cartier divisor on U_σ whose associated Weil divisor is invariant is of the form (χ^u) but

$$(\chi^u) = (\chi^{u'}) \qquad \text{if and only if} \qquad u - u' \in \sigma^\perp \cap M = M(\sigma).$$

So the T-Cartier divisors are in correspondence with $M/M(\sigma)$.

Now suppose that $X = X(F)$ is a general toric variety. Then a T-Cartier divisor is given by specifying an element $u(\sigma) \in M/M(\sigma)$, for every cone σ in F. This defines a divisor $(\chi^{-u(\sigma)})$; equivalently a fractional ideal

$$I = H^0(X, \mathcal{O}_X(D)) = A_\sigma \cdot \chi^{u(\sigma)}.$$

These maps must agree on overlaps; if τ is a face of σ then $u(\sigma) \in M/M(\sigma)$ must map to $u(\tau) \in M/M(\tau)$.

Note that it is somewhat hard to keep track of the T-Cartier divisors. We look for a way to repackage the same combinatorial data into a more convenient form. As usual, this means we should look at the dual picture.

The data

$$\{\, u(\sigma) \in M/M(\sigma) \,|\, \sigma \in F \,\},$$

for a T-Cartier divisor D determines a continuous piecewise linear function ϕ_D on the support $|F|$ of F. If $v \in \sigma$ then let

$$\phi_D(v) = \langle u(\sigma), v \rangle.$$

Compatibility of the data implies that ϕ_D is well-defined and continuous. Conversely, given any continuous function ϕ, which is linear and integral (that is, given by an element of M) on each cone, we can associate a unique T-Cartier divisor D. If $D = \sum a_i D_i$ the function is given by $\phi_D(v_i) = -a_i$, where v_i is the primitive generator of the ray corresponding to D_i.

Note that

$$\phi_D + \phi_E = \phi_{D+E} \qquad \text{and} \qquad \phi_{mD} = m\phi_D.$$

Note also that $\phi_{(\chi^u)}$ is the linear function given by $-u$. So D and E are linearly equivalent if and only if ϕ_D and ϕ_E differ by a linear function.

If X is any variety which satisfies $(*)$ then the natural map

$$\operatorname{Pic}(X) \longrightarrow \operatorname{Cl}(X),$$

is an embedding. It is an interesting to compare $\operatorname{Pic}(X)$ and $\operatorname{Cl}(X)$ on a toric variety. Denote by $\operatorname{Div}_T(X)$ the group of T-Cartier divisors.

Proposition 3.7. *Let $X = X(F)$ be the toric variety associated to a fan F which spans $N_{\mathbb{R}}$. Then there is a commutative diagram with exact rows:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & \operatorname{Div}_T(X) & \longrightarrow & \operatorname{Pic}(X) & \longrightarrow & 0 \\
 & & \| & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & \mathbb{Z}^s & \longrightarrow & \operatorname{Cl}(X) & \longrightarrow & 0.
\end{array}$$

In particular

$$\rho(X) = \operatorname{rank}(\operatorname{Pic}(X)) \leq \operatorname{rank}(\operatorname{Cl}(X)) = s - n.$$

Further $\operatorname{Pic}(X)$ *is a free abelian group.*

Proof. We have already seen that the bottom row is exact. If L is an invertible sheaf then $L|_U$ is trivial. Suppose that $L = \mathcal{O}_X(E)$. Pick a rational function such that $(f)|_U = E|_U$. Let $D = E - (f)$. Then D is T-Cartier, since it is supported away from the torus and exactness of the top row is easy.

Finally, $\operatorname{Pic}(X)$ is represented by equivalence classes of continuous, piecewise integral linear functions modulo linear functions. Clearly if $m\phi$ is linear then so is ϕ, so that $\operatorname{Pic}(X)$ is torsion free. $\square$