# 20. Cubic surfaces

The next thing to consider are cubic surfaces in $\mathbb{P}^3$.

For this we will need to work with the Grassmannian, which is the variety parametrising lines in $\mathbb{P}^3$, equivalently planes in $K^4$.

**Definition 20.1.** *The **Grassmannian** $G(k,V)$ **of** $k$**-planes in** $V$ denotes the set of all $k$ planes in $V$.*

This set is naturally a variety. We set $G(k,n) = G(k,K^n)$ and $\mathbb{G}(k,n) = G(k+1,n+1)$. The latter may be thought of as the set of $k$-planes in $\mathbb{P}^n$. The Grassmannian comes with a universal family, an incidence correspondence

$$\Sigma = \{\, (\Lambda, x) \,|\, x \in \Lambda \,\} \subset \mathbb{G}(k,n) \times \mathbb{P}^n,$$

which is a Zariski closed subset. There are two natural projections:

$$\begin{tikzcd} \Sigma \arrow[r, "q"] \arrow[d, "p"'] & \mathbb{P}^n \\ \mathbb{G}(k,n). & \end{tikzcd}$$

Using this we can deduce many of its properties. I claim that $\mathbb{G}(k,n)$ is irreducible and has dimension $(k+1)(n-k)$. We prove this by induction. Fix a point $x \in \mathbb{P}^n$. The fibre of $q$ is the set of $k$-planes containing a point. This is isomorphic to the set of $k-1$-planes in $\mathbb{P}^{n-1}$, which is irreducible of dimension $k(n-k)$. It follows that $\Sigma$ is irreducible of dimension $n + k(n-k)$. But then $\mathbb{G}(k,n)$ is certainly irreducible. If we fix $\Lambda$ then the fibre of $p$ is the set of points in $\Lambda$. This is $\Lambda$, a copy of $\mathbb{P}^k$, so that the fibres of $p$ are irreducible of dimension $k$. Thus

$$\dim \mathbb{G}(k,n) = n + k(n-k) - k = (k+1)(n-k).$$

One can use the universal family to make some interesting constructions. For example, suppose we are given a closed subset $X \subset \mathbb{P}^n$. Then $p(q^{-1}(X))$ is a closed subvariety of $\mathbb{G}(k,n)$, consisting of all $k$-planes in $\mathbb{P}^n$ which intersect $X$. The first interesting case is that of a curve $C$ in $\mathbb{P}^3$. In this case the general line does not meet the curve $C$. In fact we get a codimension one subvariety of $\mathbb{G}(1,3)$. Conversely suppose we are given a closed subvariety $\Phi$ of $\mathbb{G}(k,n)$. Then $q(p^{-1}(\Phi))$ is a closed subvariety of $\mathbb{P}^n$, equal to

$$X = \bigcup_{\Lambda \in \Phi} \Lambda.$$

Note that $X$ has the interesting property that through every point of $X$ there passes a $k$-plane. Classically such varieties are called **scrolls**. Perhaps the first interesting example of a scroll is the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$.

Let us give some more constructions of scrolls. Suppose that we are given two subvarieties $X$ and $Y$ of $\mathbb{P}^n$. Define a rational map

$$\phi\colon X \times Y \dashrightarrow \mathbb{G}(1, n),$$

by sending

$$([v], [w]) \longrightarrow [v \wedge w].$$

The subvariety in $\mathbb{P}^n$, corresponding to the image, is called the **join**. It is the closure of the union of all lines obtained by taking the span of a point of $X$ and a point of $Y$. Note that $\phi$ is a morphism if $X$ and $Y$ are disjoint and in this case we don't need to take the closure. If we take $X = Y$, then we get the **secant variety of** $X$, which is the closure of all the lines which join two points of $X$.

Suppose that we are given a morphism $f\colon X \longrightarrow Y$, with the property that there is a point $x \in X$ such that $f(x) \neq x$. Consider the morphism

$$\psi\colon X \longrightarrow \mathbb{G}(1, n),$$

which is the composition of

$$X \longrightarrow X \times Y \qquad \text{given by} \qquad x \longrightarrow (x, f(x)),$$

and the morphism $\phi$ above. As before this gives us a scroll in $\mathbb{P}^n$, by taking the image. Note that all of this generalises to products of $k$ varieties.

**Definition 20.2.** *Pick complimentary linear spaces $\Lambda_1, \Lambda_2, \dots, \Lambda_k$ of dimensions $n_1, n_2, \dots, n_k$ in $\mathbb{P}^n$, where*

$$n + 1 = \sum_i (n_i + 1).$$

*Pick rational normal curves $C_i \subset \Lambda_i$ in and pick identifications*

$$\phi_i\colon \mathbb{P}^1 \longrightarrow C_i.$$

*Let*

$$X = \bigcup_{p \in \mathbb{P}^1} \langle \phi_1(p), \phi_2(p), \dots, \phi_k(p) \rangle.$$

*Then $X$ is called a **rational normal scroll**.*

It is interesting to give some examples. Suppose that we pick two skew lines $l$ and $m$ in $\mathbb{P}^3$. Then we get a surface in $\mathbb{P}^3$, swept out by lines, meeting $l$ and $m$. Suppose we pick coordinates such that

$l = V(X, Y)$ and $m = V(Z, W)$. Identify $(0, 0, a, b)$ with $(a, b, 0, 0)$. Then it is not hard to see that we get the surface $V(XW - YZ)$.

The next case is when we take a line and a complimentary plane in $\mathbb{P}^4$. The resulting surface in $\mathbb{P}^4$ is called the cubic scroll.

Another intriguing method was proposed by Nash:

**Definition 20.3.** *Let $X \subset \mathbb{P}^N$ be a quasi-projective variety of dimension $n$. The **Gauss map** is the rational map*

$$X \dashrightarrow \mathbb{G}(n, N) \qquad \text{given by} \qquad x \longrightarrow T_x X,$$

*which sends a point to its (projective) tangent space.*

*The **Nash blow up** is given by taking the graph of this rational map.*

**Conjecture 20.4.** *We can always resolve any variety by successively taking the Nash blow up and normalising.*

Despite the very appealing nature of this conjecture (consider for example the case of curves, when we don't even need to normalise) we only know (20.4) in very special cases. The one very nice feature of the Nash blow up is that it does not involve any choices. Unfortunately it is known that one needs to normalise, and this messes up any sort of induction.

Back to cubic surfaces. The space of cubic surfaces has dimension

$$\binom{3+3}{3} - 1 = 19.$$

So the space of cubic surfaces is a copy of $\mathbb{P}^{19}$. Consider the incidence correspondence

$$\Sigma = \{\, (S, l) \,|\, l \subset S \,\} \subset \mathbb{P}^{19} \times \mathbb{G}(1, 3)$$

There are two natural projections:

$$\begin{tikzcd} \Sigma \arrow[r, "q"] \arrow[d, "p"'] & \mathbb{G}(1,3) \\ \mathbb{P}^{19} & \end{tikzcd}$$

The fibres of $q$ are the space of cubics containing a fixed line. I claim that this is a copy of $\mathbb{P}^{15}$. There are two ways to see this. Fix four points of the line. To contain any one of those points impose one linear condition. To contain all four imposes at most four conditions. But we can find a cubic containing any three points but not the fourth, so in fact they impose exactly four conditions.

Alternatively, we can always choose coordinates so that the line is $[a : b : 0 : 0]$, that is $Z = T = 0$ (using homogeneous coordinates

$[X : Y : Z : T]$). The cubic $F = 0$ contains this line if and only if the coefficients of $X^3$, $X^2Y$, $XY^2$ and $Y^3$ vanish.

Thus $\Sigma$ is irreducible and has dimension 19. We would know that every cubic contains a line provided we know that one fibre of $p$ is zero dimensional; in fact we need even less we just need one cubic and a line on the cubic which doesn't deform on the cubic. One can check that the Fermat cubic

$$X^3 + Y^3 + Z^3 + T^3 = 0,$$

has isolated lines. With a little bit more work one can show that the Fermat cubic contains exactly 27 lines. This shows that a general cubic contains at least 27 lines and every cubic contains at least one line.

Let $S$ be a smooth cubic and let $l \subset S$ be a line. It is easy to see that $l$ cannot deform, that is, $S$ is not a scroll. Consider the family of planes $\Pi$ containing $l$. The family of planes $\Pi$ containing $l$ is parametrised by $\mathbb{P}^1$. There are many ways to see this. E.g choose an auxliary skew line $m$. Then $\Pi$ intersects $m$ in a single point, which determines $\Pi$.

A plane will intersect the cubic $S$ in the line $l$ and a residual conic $C$. The conic intersects the line $l$ in two points. One can check that this conic becomes singular five times, in which case the conic is the union of two lines, neither of which are $l$. Thus there are ten lines meeting $l$. Continuing in this way, one can show that $S$ contains two skew lines $l$ and $m$. Define a rational map

$$\phi\colon l \times m \dashrightarrow S,$$

as follows. Given $(p, q) \in l \times m$ the line $\langle p, q\rangle$ will intersect $S$ in one further point $r$, at least for some open subset of $l \times m = \mathbb{P}^1 \times \mathbb{P}^1$. Send $(p, q)$ to $r$. Then $\phi$ is birational and $S$ is rational.

It is interesting to consider what happens in higher dimensions.

**Theorem 20.5** (Clemens-Griffiths)**.** *If $V \subset \mathbb{P}^4$ is a smooth cubic then $V$ is irrational.*

Starting with fourfolds the situation is quite murky. There are smooth rational cubic fourfolds. One can write down smooth cubic fourfolds which contain two skew planes. The same construction as above yields a birational map to $\mathbb{P}^2 \times \mathbb{P}^2$, which is rational. But the locus of cubic fourfolds which contain a plane in the space of all cubic fourfolds is a proper Zariski closed subset. In fact there are other special cubic fourfolds which are rational (containing ever more exotic configurations of closed subvarieties) which form a countable union of closed subvarieties. Conjecturally, however there are smooth cubic fourfolds which are irrational.