19. Projective geometry

Definition 19.1. *Let $S \subset \mathbb{P}^n$ be a set of points.*

*We say that S is in **linear general position** if any subset of $k \leq n$ points spana a $(k-1)$-plane Λ.*

Remark 19.2. *Note that if S has at least $n+1$ points then S is in linear general position if every subset of $n+1$ points spans $\mathbb{P}^n$.*

Lemma 19.3. *Any two sequences $p_0, p_1, \dots, p_{n+1}$ and $q_0, q_1, \dots, q_{n+2}$ of $n+2$ points in linear general position in $\mathbb{P}^n$ are projectively equivalent, that is, there is an element $\phi \in \operatorname{Aut}(\mathbb{P}^n) = \operatorname{PGL}(n+1)$ such that $\phi(p_i) = q_i$. Furthermore, ϕ is unique.*

Proof. Since we are just saying there is one orbit on the set of $n+2$ points in linear general position, we might as well take

$$p_0 = [1 : 0 : \dots : 0], p_1 = [0 : 1 : \dots : 0], \dots, p_n = [0 : 0 : \dots : 1], p_{n+1} = [1 : 1 : \dots : 1].$$

In terms of existence, note that ϕ corresponds to an $(n+1) \times (n+1)$ matrix with entries in K. The first $n+1$ points correspond to $n+1$ linearly independent vectors in K^{n+1}. There is a unique matrix A sending one set of vectors to the others. At this point $p_i = q_i$, $0 \leq i \leq n$ and $q_i = [a_0 : a_1 : \dots : a_n]$. Since $q_0, q_1, \dots, q_{n+2}$ are in linear general position, it follows that $a_i \neq 0$ for all i. But then the diagonal matrix with $1/a_i$ in the ith spot fixes p_j, $0 \leq j \leq n$ and takes q_{n+1} to p_{n+1}.

As for uniqueness, it is enough to show that the only ϕ which fixes the sequence $p_0, p_1, \dots, p_{n+1}$ is the identity. The fact that the corresponding matrix fixes the first $n+1$ vectors, implies that the matrix is diagonal. The fact it fixes p_{n+1} means the matrix is a scalar multiple of the identity; but then ϕ is the identity. □

In the case $n = 1$, the three standard points p_0, p_1 and p_2 correspond to 0, ∞ and 1.

A little bit of notation. We will say that a curve $C \subset \mathbb{P}^n$ is a **rational normal curve** if it is projectively equivalent to the curve

$$[S : T] \longrightarrow [S^n : S^{n-1}T : \dots : T^n].$$

Lemma 19.4. *Let $p_1, p_2, \dots, p_{n+3}$ be $n+3$ points in linear general position in $\mathbb{P}^n$.*

Then there is a unique rational normal curve $C \subset \mathbb{P}^n$ containing $p_1, p_2, \dots, p_{n+3}$.

Proof. We only prove existence. This will follow from the following way to construct rational normal curves.

Let $G(S,T)$ be a homogeneous polynomial of degree $n+1$. Then $G(S,T)$ factors,

$$G(S,T) = \prod_{i=0}^{n} (\mu_i S_i - \lambda_i T_i).$$

Assume that $G(S,T)$ has distinct roots, meaning that $[\lambda_i : \mu_i] \in \mathbb{P}^1$ are $n+1$ different points of $\mathbb{P}^1$. Consider the $n+1$ polynomials

$$G_i(S,T) = \frac{G(S,T)}{\mu_i S_i - \lambda_i T_i},$$

of degree n. Note that $G_0, G_1, \dots, G_n$ are independent in the space of polynomials of degree n; indeed if

$$\sum a_i G_i(S,T) = 0,$$

then we see that $a_i = 0$ after plugging in $[\lambda_i : \mu_i] \in \mathbb{P}^1$.

It follows that the curve C given parametrically by

$$[S : T] \longrightarrow [G_0 : G_1 : \dots : G_n],$$

is a rational normal curve.

We may rewrite this parametrisation as

$$[S : T] \longrightarrow [\frac{1}{\mu_0 S_0 - \lambda_0 T_0} : \frac{1}{\mu_1 S_1 - \lambda_1 T_1} : \dots : \frac{1}{\mu_n S_n - \lambda_n T_n}].$$

Written this way, we see that C passes through the $n+1$ coordinate points. Parametrically we send the zeroes of G to these points.

Now given any set of $n+3$ points in linear general position we have already seen that we can choose the first $n+1$ points to be the coordinate points. This leaves two more points, p_{n+2} and p_{n+3}. The image of $[1 : 0]$ is

$$[\frac{1}{\mu_0} : \frac{1}{\mu_1} : \dots : \frac{1}{\mu_n}]$$

and the image of $[0 : 1]$ is

$$[\frac{1}{\lambda_0} : \frac{1}{\lambda_1} : \dots : \frac{1}{\lambda_n}].$$

Now we can always choose p_{n+2} to be the point $[1 : 1 : \dots : 1]$, in which case we choose $\mu_i = 1$ for all i. Finally the fact that the points are in linear general position implies that the coordinates of p_{n+3} are distinct and non-zero and we can choose $\lambda_0, \lambda_1, \dots, \lambda_n$ accordingly. □

Example 19.5. *There is a simple proof of* (19.4) *in the case when $n = 2$. All rational normal curves are projectively equivalent, so in this case a rational normal curve is the same as a conic.*

In this case we want to prove that there is a unique smooth conic through any five points of $\mathbb{P}^2$, no three of which are collinear.

A conic is given by

$$aX^2 + bY^2 + cZ^2 + dYZ + eXZ + fXY,$$

and the space of all conics is naturally a copy of $\mathbb{P}^5$. The set of conics passing through a fixed point p corresponds to a hyperplane $H_p \subset \mathbb{P}^5$.

The set of conics through five points is then the intersection of five hyperplanes, which is always non-empty, so that there is always at least one conic through any five points.

If the conic is not smooth then it is either a pair of lines or a double line. A pair of lines can contain at most four points, if no three are collinear and a double line can only contain two points. So there must be a smooth conic containing the five points.

Suppose that there is more than one conic. Suppose that F and G are the defining polynomials. Then the pencil of conics given by

$$\lambda F + \mu G = 0,$$

where $[\lambda : \mu] \in \mathbb{P}^1$ also contains all five points. But any pencil of conics must contain a singular conic, and we have just seen that this is impossible.

The next natural thing to look at are quadrics $X \subset \mathbb{P}^n$, the zero sets of quadratic polynomials F. The **rank** of X is the rank of F, that is, the rank of the associated symmetric form.

Proposition 19.6. *(1) Two quadrics are projectively equivalent if and only if they have the same rank.*
(2) If X has maximal rank $n+1$ and $n > 1$ then X is rational.
(3) If X has rank less than $n+1$ and $n > 1$ then X is the cone over a quadric in $\mathbb{P}^{n-1}$.

Proof. (1) follows from the classification of symmetric bilinear forms over an algebraically closed field.

By (1) every quadric is projectively equivalent to a quadric

$$X_0^2 + X_1^2 + \cdots + X_k^2,$$

where $r = k + 1$ is the rank. If $k = n > 1$ then X is irreducible. If we pick a point p of X and project from that point then the resulting rational map

$$\pi \colon X \longrightarrow \mathbb{P}^{n-1},$$

is birational. Geometrically, we pick an auxiliary hyperplane $H \simeq \mathbb{P}^{n-1} \subset \mathbb{P}^n$ and we send a point $q \in X$ to the point $r \in \mathbb{P}^{n-1}$ where the line $\langle p, q\rangle$ meets H. Any line through p only meets X in one further point q and so π is generically one to one, and so has to be birational.

Algebraically, if $H = (X_r = 0)$ and $p = [0 : 0 : \cdots : 0 : 1]$ and we change coordinates to that F is

$$X_0^2 + X_1^2 + \cdots + X_{n-2}^2 + X_{n-1}X_n,$$

then π is the map

$$[X_0 : X_1 : \cdots : X_n] \longrightarrow [X_0 : X_1 : \cdots : X_{n-1}].$$

The inverse rational map is the map

$$[Y_0 : Y_1 : \cdots : Y_{n-1}] \longrightarrow [Y_0 : Y_1 : \cdots : Y_{n-1} : 1/Y_{n-1}(Y_0^2+Y_1^2+\cdots+Y_{n-2}^2)].$$

If $k < n$ then X is visibly a cone over quadric in the first $n - 1$ variables. □

The next thing to consider is cubics, that is, varieties defined by a single cubic polynomial.

We start with cubic curves C in $\mathbb{P}^2$. We already know that if C is smooth then C is not rational, since the genus is 1. If C is irreducible but not smooth then projection from the singular point show that C is rational, that is, birational to $\mathbb{P}^1$. In fact with a little bit more work, one can show that C is projectively equivalent either to

$$Y^2Z = X^2 + X^3 \qquad \text{or} \qquad Y^2Z = X^3,$$

a nodal cubic or a cuspidal cubic.

It is interesting to consider what happens if one projects from a point p of a smooth cubic. A general line passes through two more points q and r of the cubic and we get a double cover of $\mathbb{P}^1$. If we only get one point $q = r$ then the line through q is tangent to the curve.

It is a fact that if we choose $p \in C$ general then there are only finitely many lines through p which are tangent to C. For the cubic there are four such lines, and this gives four points in $\mathbb{P}^1$. We can always choose the first three points to to be 0, 1 and ∞ but the last point λ gives moduli.

In fact the space of all cubics is nine dimensional,

$$\binom{3+2}{2} - 1 = 9.$$

PGL(3) has dimension $3 \times 3 - 1 = 8$. So we expect a one dimensional family of non-projectively equivalent cubics.

The next thing to consider are cubic surfaces in $\mathbb{P}^3$.