18. Riemann-Roch

Let us turn to the proof of (17.4).

Definition 18.1. Let \( P(z) \in \mathbb{Q}[z] \) be a polynomial. We say that \( P(z) \) is numerical if \( P(n) \in \mathbb{Z} \) for any sufficiently large integer \( n \).

Lemma 18.2.

(1) If \( P(z) \) is a numerical polynomial then we may find integers \( c_0, c_1, \ldots, c_r \) such that
\[
P(z) = c_0 \binom{z}{r} + c_1 \binom{z}{r-1} + \cdots + c_r.
\]
In particular \( P(n) \in \mathbb{Z} \) for every \( n \in \mathbb{Z} \).

(2) If \( f: \mathbb{Z} \to \mathbb{Z} \) is any function and there is a numerical polynomial \( Q(z) \) such that \( \Delta(f) = f(n+1) - f(n) = Q(n) \) for \( n \) sufficiently large then there is a numerical polynomial \( P(z) \) such that \( f(n) = P(n) \) for \( n \) sufficiently large.

Proof. We prove (1) by induction on the degree \( r \) of \( P \). Since
\[
\binom{z}{r} = \frac{z(z-1) \cdots (z-r+1)}{r!} = \frac{z^r}{r!} + \ldots,
\]
is a polynomial of degree \( n \), they form a basis for all polynomials and we may certainly find rationals \( c_0, c_1, \ldots, c_r \) such that
\[
P(z) = c_0 \binom{z}{r} + c_1 \binom{z}{r-1} + \cdots + c_r.
\]
Note that
\[
Q(z) = \Delta P(z) = P(z+1) - P(z) = c_0 \binom{z}{r-1} + c_1 \binom{z}{r-2} + \cdots + c_r,
\]
is a numerical polynomial. By induction on the degree, \( c_0, c_1, \ldots, c_r \) are integers. It follows that \( c_r \) is an integer, as \( P(n) \) is an integer for \( n \) large. This is (1).

For (2), suppose that
\[
Q(Z) = c_0 \binom{z}{r} + c_1 \binom{z}{r-1} + \cdots + c_r,
\]
for integers \( c_0, c_1, \ldots, c_r \). Let
\[
P(z) = c_0 \binom{z}{r+1} + c_1 \binom{z}{r} + \cdots + c_r \binom{z}{1},
\]
Then \( \Delta P(z) = Q(z) \) so that \( (f - P)(n) \) is a constant \( c_{r+1} \) for any \( n \) sufficiently large, so that \( f(n) = P(n) + c_{r+1} \) for any \( n \) sufficiently large. □
Theorem 18.3 (Asymptotic Riemann-Roch). Let $X$ be a normal projective variety of dimension $n$ and let $\mathcal{O}_X(1)$ be a very ample line bundle. Suppose that $X \subset \mathbb{P}^k$ has degree $d$.

Then

$$h^0(X, \mathcal{O}_X(m)) = \frac{dm^n}{n!} + ..., $$

is a polynomial of degree $n$, for $m$ large enough, with the given leading term.

Proof. First suppose that $X$ is smooth. Let $Y$ be a general hyperplane section. Then $Y$ is smooth by Bertini. The trick is to compute $\chi(X, \mathcal{O}_X(m))$ by looking at the exact sequence

$$0 \rightarrow \mathcal{O}_X(m - 1) \rightarrow \mathcal{O}_X(m) \rightarrow \mathcal{O}_Y(m) \rightarrow 0.$$

The Euler characteristic is additive so that

$$\chi(X, \mathcal{O}_X(m)) - \chi(X, \mathcal{O}_X(m - 1)) = \chi(Y, \mathcal{O}_Y(m)).$$

(18.2) implies that $\chi(X, \mathcal{O}_X(m))$ is a polynomial of degree $n$, with the given leading term. Now apply Serre vanishing.

For the general case we need that if $X$ is normal and $Y$ is a general hyperplane section, then $Y$ is a normal projective variety of degree $d$. $Y$ is regular in codimension one by a Bertini type argument and one can check that $Y$ is $S_2$. □

We will only need (18.3) for the method of Albanese, but it is fun to use similar arguments to prove special cases of Riemann-Roch.

Theorem 18.4 (Riemann-Roch for curves). Let $C$ be a smooth projective curve of genus $g$ and let $D$ be a divisor of degree $d$.

$$h^0(X, \mathcal{O}_C(D)) = d - g + 1 + h^0(C, \mathcal{O}_C(K_C - D)).$$

Proof. We first check that

$$\chi(C, \mathcal{O}_C(D)) = d - g + 1.$$

We may write

$$D = \sum a_ip_i.$$

We proceed by induction on $\sum |a_i|$. Let $p = p_1$. If $a_1 > 0$ then consider the short exact sequence

$$0 \rightarrow \mathcal{O}_C(D - p) \rightarrow \mathcal{O}_C(D) \rightarrow \mathcal{O}_p \rightarrow 0.$$

The Euler characteristic is additive, so that

$$\chi(C, \mathcal{O}_C(D)) = \chi(C, \mathcal{O}_C(D - p)) + 1.$$
The LHS is equal to $(d - 1) - g + 1 + 1 = d - g + 1$ by induction. If $a_1 < 0$ then consider the short exact sequence

$$0 \rightarrow \mathcal{O}_C(D) \rightarrow \mathcal{O}_C(D + p) \rightarrow \mathcal{O}_p \rightarrow 0.$$ 

The Euler characteristic is additive, so that

$$\chi(C, \mathcal{O}_C(D - p)) = \chi(C, \mathcal{O}_C(D + p)) - 1.$$ 

The RHS is equal to $d - g + 1 - 1 = (d - 1) - g + 1$ by induction.

So we are reduced to the case when $d = 0$. Note that

$$h^1(C, \mathcal{O}_C(D)) = h^0(C, \mathcal{O}_C(K_C - D)),$$

by Serre duality. In particular

$$\chi(C, \mathcal{O}_C) = 1 - g,$$

which completes the induction. □

To state Riemann-Roch for surfaces, we will need intersection numbers. Suppose we work over $\mathbb{C}$. Consider the long exact sequence associated to the exponential sequence:

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O}^\text{an} X \rightarrow \mathcal{O}^*_X \rightarrow 0$$

The relevant part we are interested in is the group homomorphism:

$$c_1 : \text{Pic}(X) \rightarrow H^2(X, \mathbb{Z}).$$

Here we identified

$$H^1(X, \mathcal{O}^*_X),$$

with the group of line bundles. One can use this to define intersection numbers, using cup product of cohomology. If $\mathcal{L} = \mathcal{O}_X(L)$, so that $L$ is the divisor of zeroes and poles of a rational section of the invertible sheaf $\mathcal{L}$, we will use the notation

$$c_1(\mathcal{L})^n = L^n,$$

to denote the top self-intersection.

**Theorem 18.5** (Riemann-Roch for surfaces). Let $S$ be a smooth projective surface of irregularity $q$ and geometric genus $p_g$ over an algebraically closed field of characteristic zero. Let $D$ be a divisor on $S$.

$$\chi(S, \mathcal{O}_S(D)) = \frac{D^2}{2} - \frac{K_S \cdot D}{2} + 1 - q + p_g.$$  

**Proof.** Pick a very ample divisor $H$ such that $H + D$ is very ample. Let $C$ and $\Sigma$ be general elements of $|H|$ and $|H + D|$. Then $C$ and $\Sigma$ are smooth. There are two exact sequences

$$0 \rightarrow \mathcal{O}_S(D) \rightarrow \mathcal{O}_S(D + H) \rightarrow \mathcal{O}_C(D + H) \rightarrow 0$$
and

$$0 \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_S(D + H) \rightarrow \mathcal{O}_\Sigma(D + H) \rightarrow 0.$$ 

As the Euler characteristic is additive we have

$$\chi(S, \mathcal{O}_S(D + H)) = \chi(S, \mathcal{O}_S(D)) + \chi(C, \mathcal{O}_C(D + H))$$

$$\chi(S, \mathcal{O}_S(D + H)) = \chi(S, \mathcal{O}_S) + \chi(\Sigma, \mathcal{O}_\Sigma(D + H)).$$

Subtracting we get

$$\chi(S, \mathcal{O}_S(D)) - \chi(S, \mathcal{O}_S) = \chi(\Sigma, \mathcal{O}_\Sigma(D + H)) - \chi(C, \mathcal{O}_C(D + H)).$$

Now

$$\chi(\Sigma, \mathcal{O}_\Sigma(D + H)) = (D + H) \cdot \Sigma - \deg K_\Sigma/2$$

$$\chi(C, \mathcal{O}_C(D + H)) = (D + H) \cdot C - \deg K_C/2,$$

applying Riemann-Roch for curves to both $C$ and $\Sigma$. We have

$$(D + H) \cdot \Sigma = (D + H) \cdot H + (D + H) \cdot D,$$

and by adjunction

$$K_\Sigma = (K_S + \Sigma) \cdot \Sigma$$

and

$$K_C = (K_S + C) \cdot C.$$ 

So putting all of this together we get

$$\chi(S, \mathcal{O}_S(D)) - \chi(S, \mathcal{O}_S) = (D + H) \cdot D + \frac{1}{2}((K_S + C) \cdot C - (K_S + \Sigma) \cdot \Sigma)$$

$$= (D + H) \cdot D + \frac{1}{2}K_S \cdot (C - \Sigma) + \frac{1}{2}(H \cdot H - (H + D) \cdot (H + D))$$

$$= \frac{D \cdot D}{2} - \frac{1}{2}K_S \cdot D.$$

We have

$$c = \chi(S, \mathcal{O}_S) = h^0(S, \mathcal{O}_S) - h^1(S, \mathcal{O}_S) + h^2(S, \mathcal{O}_S) = 1 - q + p_g,$$

Here we used the highly non-trivial fact that

$$h^1(S, \mathcal{O}_S) = h^0(S, \Omega^1_S) = q,$$

from Hodge theory and Serre duality

$$h^2(S, \mathcal{O}_S) = h^0(S, \omega_S) = p_g.$$

**Definition 18.6.** Let $X$ be a quasi-projective variety and let $K$ be the function field of $X$. Let $L/K$ be a finite field extension.

The **normalisation of $X$ in $L$** is a finite morphism $Y \rightarrow X$, where $Y$ is a normal quasi-projective variety and the function field of $Y$ is $L.$
One can construct $Y$ in much the same way that one constructs the normalisation. It suffices to construct $Y$ locally, in which case we may assume that $X = \text{Spec} \ A$ is affine. In this case one simply takes $Y = \text{Spec} \ B$, where $B$ is the integral closure of $A$ inside $L$.

**Lemma 18.7.** Let $\pi: Y \rightarrow X$ be a finite morphism. If $\pi(q) = p$, then

$$\text{mult}_q Y = \deg \pi \cdot \text{mult}_p X.$$  

**Proof of (17.4).** By (18.3) we may pick $m$ sufficiently large such that if

$$\deg X_0 \subset \mathbb{P}^r$$

is the embedding given by $\mathcal{O}_X(m)$, then

$$d_0 \leq (n! + 1)(r + 1 - n).$$

By (17.3) we may find a generically finite morphism $f: X \rightarrow W$ such that either

$$\deg f \cdot \text{mult}_w W \leq n!,$$

or $W$ is a cone and

$$\deg f \leq n!.$$ 

If $W$ is a cone, then $W$ is birational to a product $\mathbb{P}^1 \times W'$. By our induction hypothesis, $W'$ is birational to a smooth projective variety $W''$. Then $W$ is birational to $W'' \times \mathbb{P}^1$. Replacing $W$ by $W'' \times \mathbb{P}^1$, we may assume that $W$ is smooth.

Let $\pi: Y \rightarrow W$ be the normalisation of $W$ in the field $L = K(X)/K(W)$. Then $Y$ is birational to $X$ and $\deg f = \deg \pi$. By (18.7),

$$\text{mult}_y Y \leq n!.$$  

$\square$