18. Riemann-Roch

Let us turn to the proof of (17.4).

Definition 18.1. *Let $P(z) \in \mathbb{Q}[z]$ be a polynomial. We say that $P(z)$ is* ***numerical*** *if $P(n) \in \mathbb{Z}$ for any sufficiently large integer n.*

Lemma 18.2.

(1) *If $P(z)$ is a numerical polynomial then we may find integers $c_0, c_1, \dots, c_r$ such that*
$$P(z) = c_0\binom{z}{r} + c_1\binom{z}{r-1} + \dots + c_r.$$
In particular $P(n) \in \mathbb{Z}$ for every $n \in \mathbb{Z}$.

(2) *If $f\colon \mathbb{Z} \longrightarrow \mathbb{Z}$ is any function and there is a numerical polynomial $Q(z)$ such that $\Delta(f) = f(n+1) - f(n) = Q(n)$ for n sufficiently large then there is a numerical polynomial $P(z)$ such that $f(n) = P(n)$ for n sufficiently large.*

Proof. We prove (1) by induction on the degree r of P. Since
$$\binom{z}{r} = \frac{z(z-1)\cdots(z-r+1)}{r!} = \frac{z^r}{r!} + \dots,$$
is a polynomial of degree n, they form a basis for all polynomials and we may certainly find rationals $c_0, c_1, \dots, c_r$ such that
$$P(z) = c_0\binom{z}{r} + c_1\binom{z}{r-1} + \dots + c_r.$$
Note that
$$Q(z) = \Delta P(z) = P(z+1) - P(z) = c_0\binom{z}{r-1} + c_1\binom{z}{r-2} + \dots + c_{r-1},$$
is a numerical polynomial. By induction on the degree, $c_0, c_1, \dots, c_{r-1}$ are integers. It follows that c_r is an integer, as $P(n)$ is an integer for n large. This is (1).

For (2), suppose that
$$Q(Z) = c_0\binom{z}{r} + c_1\binom{z}{r-1} + \dots + c_r,$$
for integers $c_0, c_1, \dots, c_r$. Let
$$P(z) = c_0\binom{z}{r+1} + c_1\binom{z}{r} + \dots + c_r\binom{z}{1}.$$
Then $\Delta P(z) = Q(z)$ so that $(f - P)(n)$ is a constant c_{r+1} for any n sufficiently large, so that $f(n) = P(n) + c_{r+1}$ for any n sufficiently large. $\square$

Theorem 18.3 (Asymptotic Riemann-Roch)**.** *Let X be a normal projective variety of dimension n and let $\mathcal{O}_X(1)$ be a very ample line bundle. Suppose that $X \subset \mathbb{P}^k$ has degree d.*

Then
$$h^0(X, \mathcal{O}_X(m)) = \frac{dm^n}{n!} + ...,$$
is a polynomial of degree n, for m large enough, with the given leading term.

Proof. First suppose that X is smooth. Let Y be a general hyperplane section. Then Y is smooth by Bertini. The trick is to compute $\chi(X, \mathcal{O}_X(m))$ by looking at the exact sequence
$$0 \longrightarrow \mathcal{O}_X(m-1) \longrightarrow \mathcal{O}_X(m) \longrightarrow \mathcal{O}_Y(m) \longrightarrow 0.$$
The Euler characteristic is additive so that
$$\chi(X, \mathcal{O}_X(m)) - \chi(X, \mathcal{O}_X(m-1)) = \chi(Y, \mathcal{O}_Y(m)).$$
(18.2) implies that $\chi(X, \mathcal{O}_X(m))$ is a polynomial of degree n, with the given leading term. Now apply Serre vanishing.

For the general case we need that if X is normal and Y is a general hyperplane section, then Y is a normal projective variety of degree d. Y is regular in codimension one by a Bertini type argument and one can check that Y is S_2. □

We will only need (18.3) for the method of Albanese, but it is fun to use similar arguments to prove special cases of Riemann-Roch.

Theorem 18.4 (Riemann-Roch for curves)**.** *Let C be a smooth projective curve of genus g and let D be a divisor of degree d.*
$$h^0(X, \mathcal{O}_C(D)) = d - g + 1 + h^0(C, \mathcal{O}_C(K_C - D)).$$

Proof. We first check that
$$\chi(C, \mathcal{O}_C(D)) = d - g + 1.$$
We may write
$$D = \sum a_i p_i.$$
We proceed by induction on $\sum |a_i|$. Let $p = p_1$. If $a_1 > 0$ then consider the short exact sequence
$$0 \longrightarrow \mathcal{O}_C(D-p) \longrightarrow \mathcal{O}_C(D) \longrightarrow \mathcal{O}_p \longrightarrow 0.$$
The Euler characteristic is additive, so that
$$\chi(C, \mathcal{O}_C(D)) = \chi(C, \mathcal{O}_C(D-p)) + 1.$$

The LHS is equal to $(d-1)-g+1+1 = d-g+1$ by induction. If $a_1 < 0$ then consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_C(D) \longrightarrow \mathcal{O}_C(D+p) \longrightarrow \mathcal{O}_p \longrightarrow 0.$$

The Euler characteristic is additive, so that

$$\chi(C, \mathcal{O}_C(D-p)) = \chi(C, \mathcal{O}_C(D+p)) - 1.$$

The RHS is equal to $d-g+1-1 = (d-1)-g+1$ by induction.

So we are reduced to the case when $d = 0$. Note that

$$h^1(C, \mathcal{O}_C(D)) = h^0(C, \mathcal{O}_C(K_C - D)),$$

by Serre duality. In particular

$$\chi(C, \mathcal{O}_C) = 1 - g,$$

which completes the induction. □

To state Riemann-Roch for surfaces, we will need intersection numbers. Suppose we work over $\mathbb{C}$. Consider the long exact sequence associated to the exponential sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X^{\mathrm{an}} \longrightarrow \mathcal{O}_X^* \longrightarrow 0$$

The relevant part we are interested in is the group homomorphism:

$$c_1 \colon \operatorname{Pic}(X) \longrightarrow H^2(X, \mathbb{Z}).$$

Here we identified

$$H^1(X, \mathcal{O}_X^*),$$

with the group of line bundles. One can use this to define intersection numbers, using cup product of cohomology. If $\mathcal{L} = \mathcal{O}_X(L)$, so that L is the divisor of zeroes and poles of a rational section of the invertible sheaf $\mathcal{L}$, we will use the notation

$$c_1(\mathcal{L})^n = L^n,$$

to denote the top self-intersection.

Theorem 18.5 (Riemann-Roch for surfaces)**.** *Let S be a smooth projective surface of irregularity q and geometric genus p_g over an algebraically closed field of characteristic zero. Let D be a divisor on S.*

$$\chi(S, \mathcal{O}_S(D)) = \frac{D^2}{2} - \frac{K_S \cdot D}{2} + 1 - q + p_g.$$

Proof. Pick a very ample divisor H such that $H+D$ is very ample. Let C and Σ be general elements of $|H|$ and $|H+D|$. Then C and Σ are smooth. There are two exact sequences

$$0 \longrightarrow \mathcal{O}_S(D) \longrightarrow \mathcal{O}_S(D+H) \longrightarrow \mathcal{O}_C(D+H) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(D+H) \longrightarrow \mathcal{O}_\Sigma(D+H) \longrightarrow 0.$$

As the Euler characteristic is additive we have

$$\begin{aligned}\chi(S, \mathcal{O}_S(D+H)) &= \chi(S, \mathcal{O}_S(D)) + \chi(C, \mathcal{O}_C(D+H))\\ \chi(S, \mathcal{O}_S(D+H)) &= \chi(S, \mathcal{O}_S) + \chi(\Sigma, \mathcal{O}_\Sigma(D+H)).\end{aligned}$$

Subtracting we get

$$\chi(S, \mathcal{O}_S(D)) - \chi(S, \mathcal{O}_S) = \chi(\Sigma, \mathcal{O}_\Sigma(D+H)) - \chi(C, \mathcal{O}_C(D+H)).$$

Now

$$\begin{aligned}\chi(\Sigma, \mathcal{O}_\Sigma(D+H)) &= (D+H)\cdot\Sigma - \deg K_\Sigma/2\\ \chi(C, \mathcal{O}_C(D+H)) &= (D+H)\cdot C - \deg K_C/2,\end{aligned}$$

applying Riemann-Roch for curves to both C and Σ. We have

$$(D+H)\cdot\Sigma = (D+H)\cdot H + (D+H)\cdot D,$$

and by adjunction

$$K_\Sigma = (K_S+\Sigma)\cdot\Sigma \qquad \text{and} \qquad K_C = (K_S+C)\cdot C.$$

So putting all of this together we get

$$\begin{aligned}\chi(S, \mathcal{O}_S(D)) - \chi(S, \mathcal{O}_S) &= (D+H)\cdot D + \frac{1}{2}((K_S+C)\cdot C - (K_S+\Sigma)\cdot\Sigma)\\ &= (D+H)\cdot D + \frac{1}{2}K_S\cdot(C-\Sigma) + \frac{1}{2}(H\cdot H - (H+D)\cdot(H+D))\\ &= \frac{D\cdot D}{2} - \frac{1}{2}K_S\cdot D.\end{aligned}$$

We have

$$c = \chi(S, \mathcal{O}_S) = h^0(S, \mathcal{O}_S) - h^1(S, \mathcal{O}_S) + h^2(S, \mathcal{O}_S) = 1 - q + p_g.$$

Here we used the highly non-trivial fact that

$$h^1(S, \mathcal{O}_S) = h^0(S, \Omega^1_S) = q,$$

from Hodge theory and Serre duality

$$h^2(S, \mathcal{O}_S) = h^0(S, \omega_S) = p_g. \qquad \square$$

Definition 18.6. *Let X be a quasi-projective variety and let K be the function field of X. Let L/K be a finite field extension.*

*The **normalisation of X in L** is a finite morphism $Y \longrightarrow X$, where Y is a normal quasi-projective variety and the function field of Y is L.*

One can consruct Y in much the same way that one constructs the normalisation. It suffices to construct Y locally, in which case we may assume that $X = \operatorname{Spec} A$ is affine. In this case one simply takes $Y = \operatorname{Spec} B$, where B is the integral closure of A inside L.

Lemma 18.7. *Let $\pi \colon Y \longrightarrow X$ be a finite morphism.*
If $\pi(q) = p$, then

$$\operatorname{mult}_q Y = \deg \pi \cdot \operatorname{mult}_p X.$$

Proof of (17.4). By (18.3) we may pick m sufficiently large such that if

$$\deg X_0 \subset \mathbb{P}^r$$

is the embedding given by $\mathcal{O}_X(m)$, then

$$d_0 \leq (n! + 1)(r + 1 - n).$$

By (17.3) we may find a generically finite morphism $f \colon X \dashrightarrow W$ such that either

$$\deg f \operatorname{mult}_w W \leq n!,$$

or W is a cone and

$$\deg f \leq n!.$$

If W is a cone, then W is birational to a product $\mathbb{P}^1 \times W'$. By our induction hypothesis, W' is birational to a smooth projective variety W''. Then W is birational to $W'' \times \mathbb{P}^1$. Replacing W by $W'' \times \mathbb{P}^1$, we may assume that W is smooth.

Let $\pi \colon Y \longrightarrow W$ be the normalisation of W in the field $L = K(X)/K(W)$. Then Y is birational to X and $\deg f = \deg \pi$. By (18.7),

$$\operatorname{mult}_y Y \leq n!. \qquad \square$$