17. The Albanese method

Beyond the dimension of the Zariski tangent space, perhaps the most basic invariant of any singular point is:

Definition 17.1. *Let $X \subset M$ be a subvariety of a smooth variety. The* ***multiplicity of*** *X* ***at*** *$p \in M$ is the largest μ such that $\mathcal{I}_p \subset \mathfrak{m}^\mu$ where $\mathfrak{m}$ is the maximal ideal of M at p in $\mathcal{O}_{M,p}$ and $\mathcal{I}$ is the ideal sheaf of X in M.*

Example 17.2. *Let $X \subset \mathbb{A}^{n+1}$ be defined by a single equation*

$$f(x_1, x_2, \dots, x_n) = 0.$$

The multiplicity of X at the origin is the degree of as a power series, that is, the smallest degree of a monomial which appears in f.

The multiplicity has two basic properties. X is smooth at p if and only if the multiplicity is one and the multiplicity is upper semi-continuous in families.

We next describe the method of Albanese. Start with $X \subset \mathbb{P}^n$. Now re-embed X by the very ample line bundle $\mathcal{O}_X(m)$, where m is very large, so that $X = X_0 \subset \mathbb{P}^r$, where r is large. Pick a point $p = p_0 \in X_0$, where the multiplicity is largest, to get $X_1 \subset \mathbb{P}^{r-1}$. Now pick a point $p_1 \in X_1$ of largest multiplicity and project down to get $X_2 \subset \mathbb{P}^{r-2}$. Continuing in this way, always projecting from a point of maximal mulitplicity, we construct $X_i \subset \mathbb{P}^{r-i}$.

Theorem 17.3. *If*

$$\deg X_0 < (n! + 1)(r + 1 - n),$$

then the Albanese algorithm stops with a variety X_k and a generically finite map $f_k \colon X_0 \dashrightarrow X_k$, such that either

(1) $\deg f_k \operatorname{mult}_p(X_k) \leq n!$, *or*
(2) X_k *is a cone and* $\deg f_k \leq n!$.

Corollary 17.4. *Assume that every variety of dimension at most $n-1$ is birational to a smooth projective variety.*

Then every projective variety is birational to a projective variety with singularities of multiplicity at most $n!$.

Note that this resolves singularities for curves, since $1! = 1$ and a point of multiplicity one is a smooth point of X. Even for surfaces we get down to points of multiplicity two, which are not so bad. Starting with threefolds, the situation is not nearly so rosy, especially when one realises that if f is a hypersurface singularity of arbitrary multiplicity, then the suspension of f, $x^2 + f$, is a hypersurface singularity of multiplicity two. It is pretty clear that resolving $x^2 + f$ entails resolving f.

Unfortunately it seems impossible to improve the bound given in (17.3).

We will need:

Theorem 17.5. *Let $X \subset \mathbb{P}^r$ be an irreducible projective variety of degree d and dimension n.*

If X is not contained in a hyperplane, then

$$d \geq r + 1 - n.$$

Proof of (17.3). We will prove by induction on k that if after the first k steps we don't have a cone and (1) never holds, that

$$\deg f_k \cdot \operatorname{mult}_p(X_k) \leq (n! + 1)(r - k + 1 - n).$$

Suppose that p is a point of maximal multiplicity μ. If X_k is a cone with vertex p, then there is nothing to prove. Otherwise let X_{k+1} be the closure of the image of p under projection, and let $\pi \colon X_k \dashrightarrow X_{k+1}$ be the resulting rational map. As X_k is not a cone over p, π is generically finite.

Let d_k be the degree of X_k. The degree of π is the number of times a general line through p and another point of X_k meets X_k outside p. The degree d_{k+1} of X_{k+1} is the number of points a general space Λ of dimension $n + k + 1 - r$ will meet $X_{k+1} \subset \mathbb{P}^{r-k-1}$. Let $\Lambda' = \langle \Lambda, p \rangle$ be the span of Λ and p. This will meet X_{k+1} in $d_k - \mu$ points, other than p. So, we have

$$\deg \pi \cdot d_{k+1} = d_k - \mu.$$

If

$$\deg f_k \cdot \mu > n!,$$

then

$$\begin{aligned}
\deg f_{k+1} \cdot d_{k+1} &= \deg f_k \deg \pi \cdot d_{k+1} \\
&= \deg f_k \cdot d_k - \deg f_k \mu \\
&\leq \deg f_k \cdot d_k - (n! + 1) \\
&\leq (n! + 1)(r - k + 1 - n) - (n! + 1) \\
&\leq (n! + 1)(r - (k + 1) + 1 - n).
\end{aligned}$$

This completes the induction.

It follows that eventually either

$$\deg f_k \cdot \operatorname{mult}_p X_k \leq n!,$$

which is case (1) or we get a cone (in the extreme case when $k = r - n$, so that $X_k = \mathbb{P}^n$ then we have a cone, since $\mathbb{P}^n$ is cone).

As $X_k \subset \mathbb{P}^{r-k}$ is not contained in a hyperplane, we have

$$d_k \geq (r - k + 1 - n).$$

It follows that if X_k is a cone, then

$$\deg f_k \leq n!. \qquad \square$$

Notice how truly bizarre this argument is; presumably projecting from a point will introduce all sorts of bad singularities (corresponding to secant lines and so on), but just by projecting from the point of maximal multiplicity works.

Example 17.6. *Let*

$$m_1 \leq m_2 \leq \dots \leq m_r,$$

be a sequence of positive integers. Let C be the image of

$$t \longrightarrow (t^{m_1}, t^{m_2}, t^{m_3}, \dots, t^{m_r}),$$

inside $\mathbb{A}^r$. If we project from $(1, 0, 0, \dots, 0)$, then we get the image of

$$t \longrightarrow (t^{m_2 - m_1}, t_2^{m_3 - m_1}, t_3^{m_4 - m_1}, \dots, t_r^{m_r - m_1}),$$

inside $\mathbb{A}^{r-1}$. It is intuitively clear that the projection of C is less singular than C, but it is hard to say exactly why; for example the multiplicity might go up.