16. The normalisation

We start to consider the problem of resolution of singularities. There are two extremes of what one could hope to prove. Here is the weakest statement one could hope for:

Conjecture 16.1 (Resolution of singularities, weak form)**.** *Given a finitely generated field extension K/k there is always a smooth projective variety X over k with function field K.*

Here is essentially the strongest form:

Conjecture 16.2 (Resolution of singularities, strong form)**.** *Let X be a variety over a field k. Then we may find a projective birational morphism $\pi\colon Y \longrightarrow X$ where*

(1) Y is smooth,
(2) π is an isomorphism outside the singular locus of X,
(3) the locus where π is not an isomorphism is a divisor E in Y,
(4) there is a divisor A on Y supported on E which is ample over X,
(5) every component of E is smooth and the tangent spaces to each component intersect in the expected dimension, and
(6) π is invariant under automorphisms of X.

Over $\mathbb{C}$ (5) is equivalent to requiring that E look like the coordinate axes locally analytically (that is, locally in the Euclidean topology not just the Zariski topology). For (6) we actually require invariance under local analytic isomorphism and even invariance under the Galois group of a field extension. More about this later.

In practice, with our current understanding of the problem, if one can prove (16.1) then the same methods can be pushed to prove some form of (16.2).

There are quite a few interesting geometric and algebraic approaches to resolution of singularities and in this section we review some of them. Even though these methods don't always work, they introduce ideas and techniques which are of considerable independent interest.

Definition 16.3. *Let X be an integral scheme. We say that X is* ***normal*** *if all of the local rings $\mathcal{O}_{X,p}$ are integrally closed.*

The ***normalisation of*** *X is a morphism $Y \longrightarrow X$ from a normal scheme, which is universal amongst all such morphisms. If $Z \longrightarrow X$ is a morphism from a normal scheme Z, then there is a unique morphism*

$Z \longrightarrow Y$ which make the diagram commute:

$$\begin{tikzcd} Z \arrow[r] \arrow[dr] & Y \arrow[d] \\ & X. \end{tikzcd}$$

One can always construct the normalisation of a scheme as follows. By the universal property, it suffices to construct the normalisation locally. If $X = \operatorname{Spec} A$, then $Y = \operatorname{Spec} B$, where B is the integral closure of A inside the field of fractions. Note that if X is quasi-projective variety then the normalisation $Y \longrightarrow X$ is a finite and birational morphism.

Definition 16.4. *Let X be a scheme. We say that X satisfies **condition** S_2 if every regular function defined on an open subset U whose complement has codimension at least two, extends to the whole of X.*

Lemma 16.5 (Serre's criterion)**.** *Let X be an integral scheme.*

Then X is normal if and only if it is regular in codimension one (condition R_1) and satisfies condition S_2.

Note that this gives a simple method to resolve singularities of curves. If C is a curve, the normalisation $C' \longrightarrow C$ is smooth in codimension one, which is to say that C' is smooth.

Note that lots of surface singularities are normal. For example, every hypersurface singularity is S_2, so that a hypersurface singularity is normal if and only if it is smooth in codimension one. Similarly, every quotient singularity is normal.

Before we pass on to other methods, it is interesting to write down some example of varieties which are R_1 but not normal, that is, which are not S_2.

Example 16.6. *Let S be the union of two smooth surfaces S_1 and S_2 joined at a single point p. For example, two general planes in $\mathbb{A}^4$ which both contain the same point p. Let $U = S - \{p\}$. Then U is the disjoint union of $U_1 = S_1 - \{p\}$ and $S_2 - \{p\}$, so U is smooth and the codimension of the complement is two. Let $f \colon U \longrightarrow k$ be the function which takes the value 1 on U_1 and the value 0 on U_2. Then f is regular, but it does not even extend to a continuous function, let alone a regular function, on S.*

Let C be a projection of a rational normal quartic down to $\mathbb{P}^3$, for example the image of

$$[S : T] \longrightarrow [S^4 : S^3T : ST^3 : T^4] = [A : B : C : D].$$

Let S be the cone over C. Then S is regular in codimension one, but it is not S_2. Indeed,

$$\frac{B^2}{A} = S^2T^2 = \frac{C^2}{D}.$$

is a regular function whose only pole is along $A = 0$ and $D = 0$, that is, only at $(0,0,0,0)$ of S.

Note that the coordinate ring

$$k[S^4, S^3T, ST^3, T^4] = \frac{k[A,B,C,D]}{\langle AD - BC, B^3 - A^2C, C^3 - BD^2\rangle},$$

is indeed not integrally closed in its field of fractions. Indeed,

$$\alpha = \frac{B^2}{A},$$

is a root of the monic polynomial $u^2 - BC$.