15. Cohomology of projective space

Let us calculate the cohomology of projective space.

Theorem 15.1. *Let A be a Noetherian ring. Let $X = \mathbb{P}^r_A$.*

(1) The natural map $S \longrightarrow \Gamma_*(X, \mathcal{O}_X)$ is an isomorphism.

(2)

$$H^i(X, \mathcal{O}_X(n)) = 0 \qquad \text{for all} \qquad 0 < i < r \text{ and } n.$$

(3)

$$H^r(X, \mathcal{O}_X(-r-1)) \simeq A.$$

(4) The natural map

$$H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-n-r-1)) \longrightarrow H^r(X, \mathcal{O}_X(-r-1)) \simeq A,$$

is a perfect pairing of finitely generated free A-modules.

Proof. Let

$$\mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X(n).$$

Then $\mathcal{F}$ is a quasi-coherent sheaf. Let $\mathcal{U}$ be the standard open affine cover. As every intersection is affine, it follows that we may compute sheaf cohomology using this cover. Now

$$\Gamma(U_I, \mathcal{F}) = S_{x_I},$$

where

$$x_I = \prod_{i \in I} x_i.$$

Thus Čech cohomology is the cohomology of the complex

$$\prod_{i=0}^{r} S_{x_i} \longrightarrow \prod_{i<j}^{r} S_{x_i x_j} \longrightarrow \dots \longrightarrow S_{x_0 x_1, \dots x_r}.$$

The kernel of the first map is just $H^0(X, \mathcal{F})$, which we already know is S. Now let us turn to $H^r(X, \mathcal{F})$. It is the cokernel of the map

$$\prod_i S_{x_0 x_1 \dots \hat{x}_i \dots x_r} \longrightarrow S_{x_0 x_1 \dots x_r}.$$

The last term is the free A-module with generators all monomials in the Laurent ring (that is, we allow both positive and negative powers).

The image is the set of monomials where x_i has non-negative exponent for at least one i. Thus the cokernel is naturally identified with the free A-module generated by arbitrary products of reciprocals x_i^{-1},

$$\{\, x_0^{l_0} x_1^{l_1} \dots x_r^{l_r} \mid l_i < 0 \,\}.$$

The grading is then given by
$$l = \sum_{i=0}^{r} l_i.$$
In particular
$$H^r(X, \mathcal{O}_X(-r-1)),$$
is the free A-module with generator $x_0^{-1}x_1^{-1}\dots x_r^{-1}$. Hence (3).

To define a pairing, we declare
$$x_0^{l_0}x_1^{l_1}\dots x_r^{l_r},$$
to be the dual of
$$x_0^{m_0}x_1^{m_1}\dots x_r^{m_r} = x_0^{-1-l_0}x_1^{-1-l_1}\dots x_r^{-1-l_r}.$$
As $m_i \geq 0$ if and only if $l_i < 0$ it is straightforward to check that this gives a perfect pairing. Hence (4).

It remains to prove (2). If we localise the complex above with respect to x_r, we get a complex which computes $\mathcal{F}|_{U_r}$, which is zero in positive degree, as U_r is affine. Thus
$$H^i(X, \mathcal{F})_{x_r} = 0,$$
for $i > 0$ so that every element of $H^i(X, \mathcal{F})$ is annihilated by some power of x_r.

To finish the proof, we will show that multiplication by x_r induces an inclusion of cohomology. We proceed by induction on the dimension. Suppose that $r > 1$ and let $Y \simeq \mathbb{P}^{r-1}_A$ be the hyperplane $x_r = 0$. Then
$$\mathcal{I}_Y = \mathcal{O}_X(-Y) = \mathcal{O}_X(-1).$$
Thus there are short exact sequences
$$0 \longrightarrow \mathcal{O}_X(n-1) \longrightarrow \mathcal{O}_X(n) \longrightarrow \mathcal{O}_Y(n) \longrightarrow 0.$$
Now $H^i(Y, \mathcal{O}_Y(n)) = 0$ for $0 < i < r-1$, by induction, and the natural restriction map
$$H^0(X, \mathcal{O}_X(n)) \longrightarrow H^0(Y, \mathcal{O}_Y(n)),$$
is surjective (every polynomial of degree n on Y is the restriction of a polynomial of degree n on X). Thus
$$H^i(X, \mathcal{O}_X(n-1)) \simeq H^i(X, \mathcal{O}_X(n)),$$
for $0 < i < r-1$, and even if $i = r-1$, then we get an injective map. But this map is the one induced by multiplication by x_r. $\square$

Theorem 15.2 (Serre vanishing)**.** *Let X be a projective variety over a Noetherian ring and let $\mathcal{O}_X(1)$ be a very ample line bundle on X. Let $\mathcal{F}$ be a coherent sheaf.*

(1) $H^i(X, \mathcal{F})$ *are finitely generated A-modules.*
(2) *There is an integer n_0 such that $H^i(X, \mathcal{F}(n)) = 0$ for all $n \geq n_0$ and $i > 0$.*

Proof. By assumption there is an immersion $i\colon X \longrightarrow \mathbb{P}^r_A$ such that $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}^r_A}(1)$. As X is projective, it is proper and so i is a closed immersion. If $\mathcal{G} = i_*\mathcal{F}$ then

$$H^i(\mathbb{P}^r_A, \mathcal{G}) \simeq H^i(X, \mathcal{F}).$$

Replacing X by $\mathbb{P}^r_A$ and $\mathcal{F}$ by $\mathcal{G}$ we may assume that $X = \mathbb{P}^r_A$.

If $\mathcal{F} = \mathcal{O}_X(q)$ then the result is given by (15.1). Thus the result also holds is $\mathcal{F}$ is a direct sum of invertible sheaves. The general case proceeds by descending induction on i. Now

$$H^i(X, \mathcal{F}) = 0,$$

if $i > r$, by Grothendieck's vanishing theorem. On the other hand, $\mathcal{F}$ is a quotient of a direct sum $\mathcal{E}$ of invertible sheaves. Thus there is an exact sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\mathcal{R}$ is coherent. Twisting by $\mathcal{O}_X(n)$ we get

$$0 \longrightarrow \mathcal{R}(n) \longrightarrow \mathcal{E}(n) \longrightarrow \mathcal{F}(n) \longrightarrow 0.$$

Taking the long exact sequence of cohomology, we get isomorphisms

$$H^i(X, \mathcal{F}(n)) \simeq H^{i+1}(X, \mathcal{R}(n)),$$

for n large enough, and we are done by descending induction on i. □

Theorem 15.3. *Let A be a Noetherian ring and let X be a proper scheme over A. Let $\mathcal{L}$ be an invertible sheaf on X. TFAE*

(1) *$\mathcal{L}$ is ample.*
(2) *For every coherent sheaf $\mathcal{F}$ on X there is an integer n_0 such that*

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0,$$

for $n > n_0$.

Proof. Suppose that (1) holds. Pick a positive integer m such that $\mathcal{M} = L^{\otimes m}$ is very ample. Let $\mathcal{F}_r = \mathcal{F} \otimes \mathcal{L}^r$, for $0 \leq r \leq m-1$. By (15.2) we may find n_r depending on r such that $H^i(X, \mathcal{F}_r \otimes \mathcal{M}^n) = 0$ for all $n > n_r$ and $i > 0$. Let p be the maximum of the n_r. Given $n > n_0 = pm$, we may write $n = qm + r$, for some $0 \leq r \leq m-1$ and $q > p$. Then

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = H^i(X, \mathcal{F}_i \otimes \mathcal{M}^q) = 0,$$

for any $i > 0$. Hence (1) implies (2).

Now suppose that (2) holds. Let $\mathcal{F}$ be a coherent sheaf. Let $p \in X$ be a closed point. Consider the short exact sequence

$$0 \longrightarrow \mathcal{I}_p\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F} \otimes \mathcal{O}_p \longrightarrow 0,$$

where $\mathcal{I}_p$ is the ideal sheaf of p. If we tensor this exact sequence with $\mathcal{L}^n$ we get an exact sequence

$$0 \longrightarrow \mathcal{I}_p\mathcal{F} \otimes \mathcal{L}^n \longrightarrow \mathcal{F} \otimes \mathcal{L}^n \longrightarrow \mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{O}_p \longrightarrow 0.$$

By hypotheses we can find n_0 such that

$$H^1(X, \mathcal{I}_p\mathcal{F} \otimes \mathcal{L}^n) = 0,$$

for all $n \geq n_0$. It follows that the natural map

$$H^0(X, \mathcal{F} \otimes \mathcal{L}^n) \longrightarrow H^0(X, \mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{O}_p),$$

is surjective, for all $n \geq n_0$. It follows by Nakayama's lemma applied to the local ring $\mathcal{O}_{X,p}$ that that the stalk of $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections. As $\mathcal{F}$ is a coherent sheaf, for each integer $n \neq n_0$ there is an open subset U, depending on n, such that sections of $H^0(X, \mathcal{F} \otimes \mathcal{L}^n)$ generate the sheaf at every point of U.

If we take $\mathcal{L} = \mathcal{O}_X$ it follows that there is an integer n_1 such that $\mathcal{L}^{n_1}$ is generated by global sections over an open neighbourhood V of p. For each $0 \leq r \leq n_1 - 1$ we may find U_r such that $\mathcal{F} \otimes \mathcal{L}^{n_0+r}$ is generated by global sections over U_r. Now let

$$U_p = V \cap U_0 \cap U_1 \cap \cdots \cap U_{n_1-1}.$$

Then

$$\mathcal{F} \otimes \mathcal{L}^n = (\mathcal{F} \otimes \mathcal{L}^{n_0+r}) \otimes (\mathcal{L}^{n_1})^m,$$

is generated by global sections over the whole of U_p for all $n \neq n_0$.

Now use compactness of X to conclude that we can cover X by finitely many U_p. $\square$

Theorem 15.4 (Serre duality)**.** *Let X be a smooth projective variety of dimension n over an algebraically closed field. Then there is an invertible sheaf ω_X such that*

(1) $h^n(X, \omega_X) = 1$.
(2) Given any other invertible sheaf $\mathcal{L}$ there is a perfect pairing

$$H^i(X, \mathcal{L}) \times H^{n-i}(X, \omega_X \otimes \mathcal{L}^*) \longrightarrow H^n(X, \omega_X).$$

Example 15.5. *Let $X = \mathbb{P}^r_k$. Then $\omega_X = \mathcal{O}_X(-r-1)$ is a dualising sheaf.*

In fact, on any smooth projective variety, the dualising sheaf is precisely the canonical sheaf. This expresses a remarkable coincidence between the dualising sheaf, which is something defined in terms of sheaf cohomology and the determinant of the sheaf of Kähler differentials, which is something which comes from calculus on the variety.

Theorem 15.6. *Let $X = X(F)$ be a toric variety over $\mathbb{C}$ and let D be a T-Cartier divisor. Given $u \in M$ let*

$$Z(u) = \{\, v \in |F| \mid \langle u, v\rangle \geq \psi_D(v) \,\}.$$

Then

$$H^p(X, \mathcal{O}_X(D)) = \bigoplus_{u\in M} H^p(X, \mathcal{O}_X(D))_u \qquad \textit{where} \qquad H^p(X, \mathcal{O}_X(D))_u = H^p_{Z(u)}(|F|).$$

Some explanation is in order. Note that the cohomology groups of X are naturally graded by M. (15.6) identifies the graded pieces.

$$H^p_{Z(u)}(|F|) = H^p(|F|, |F| - Z(u), \mathbb{C}).$$

denotes local cohomology. This comes with a long exact sequence for the pair. If X is an affine toric variety then both $|F|$ and $Z(u)$ are convex and the local cohomology vanishes. More generally, if D is ample, then then both $|F|$ and $Z(u)$ are convex and the local cohomology vanishes. This gives a slightly stronger result than Serre vanishing in the case of an arbitrary variety.