14. Čech Cohomology

We would like to have a way to compute sheaf cohomology. Let X be a topological space and let $\mathcal{U} = \{U_i\}$ be an open cover, which is locally finite. The group of k-cochains is

$$C^k(\mathcal{U}, \mathcal{F}) = \bigoplus_I \Gamma(U_I, \mathcal{F}),$$

where I runs over all $(k+1)$-tuples of indices and

$$U_I = \bigcap_{i \in I} U_i,$$

denotes intersection. k-cochains are skew-commutative, so that if we switch two indices we get a sign change.

Define a coboundary map

$$\delta^k \colon C^k(\mathcal{U}, \mathcal{F}) \longrightarrow C^{k+1}(\mathcal{U}, \mathcal{F}).$$

Given $\sigma = (\sigma_I)$, we have to construct $\tau = \delta(\sigma) \in C^{k+1}(\mathcal{U}, \mathcal{F})$. We just need to determine the components τ_J of τ. Now $J = \{j_0, j_1, \dots, j_k\}$. If we drop an index, then we get a k-tuple. We define

$$\tau_J = \left(\sum_{i=0}^{k} (-1)^i \sigma_{J - \{i_i\}} \right) \Bigg|_{U_J}.$$

The key point is that $\delta^2 = 0$. So we can take cohomology

$$\check{H}^i(\mathcal{U}, \mathcal{F}) = Z^i(\mathcal{U}, \mathcal{F}) / B^i(\mathcal{U}, \mathcal{F}).$$

Here Z^i denotes the group of i-cocycles, those elements killed by δ^i and B^i denotes the group of coboundaries, those cochains which are in the image of δ^{i-1}. Note that $\delta^i(B^i) = \delta^i \delta^{i-1}(C^{i-1}) = 0$, so that $B^i \subset Z^i$.

The problem is that this is not enough. Perhaps our open cover is not fine enough to capture all the interesting cohomology. A **refinement** of the open cover $\mathcal{U}$ is an open cover $\mathcal{V}$, together with a map h between the indexing sets, such that if V_j is an open subset of the refinement, then for the index $i = h(j)$, we have $V_j \subset U_i$. It is straightforward to check that there are maps,

$$\check{H}^i(\mathcal{U}, \mathcal{F}) \longrightarrow \check{H}^i(\mathcal{V}, \mathcal{F}),$$

on cohomology. Taking the (direct) limit, we get the Čech cohomology groups,

$$\check{H}^i(X, \mathcal{F}).$$

For example, consider the case $i = 0$. Given a cover, a cochain is just a collection of sections, (σ_i), $\sigma_i \in \Gamma(U_i, \mathcal{F})$. This cochain is a cocycle if $(\sigma_i - \sigma_j)|_{U_{ij}} = 0$ for every i and j. By the sheaf axiom, this means that there is a global section $\sigma \in \Gamma(X, \mathcal{F})$, so that in fact

$$H^0(\mathcal{U}, \mathcal{F}) = \Gamma(X, \mathcal{F}).$$

It is also sometimes possible to untwist the definition of $\check{H}^1$. A 1-cocycle is precisely the data of a collection

$$(\sigma_{ij}) \in \Gamma(\mathcal{U}, \mathcal{F}),$$

such that

$$\sigma_{ij} - \sigma_{ik} + \sigma_{jk} = 0.$$

In general of course, one does not want to compute these things using limits. The question is how fine does the cover have to be to compute the cohomology? As a first guess one might require that

$$\check{H}^i(U_j, \mathcal{F}) = 0,$$

for all j, and $i > 0$. In other words there is no cohomology on each open subset. But this is not enough. One needs instead the slightly stronger condition that

$$\check{H}^i(U_I, \mathcal{F}) = 0.$$

Theorem 14.1 (Leray). *If X is a topological space and $\mathcal{F}$ is a sheaf of abelian groups and $\mathcal{U}$ is an open cover such that*

$$\check{H}^i(U_I, \mathcal{F}) = 0,$$

for all $i > 0$ and indices I, then in fact the natural map

$$\check{H}^i(\mathcal{U}, \mathcal{F}) \simeq \check{H}^i(X, \mathcal{F}),$$

is an isomorphism.

Finally, we need to construct the coboundary maps. Suppose that we are given a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0.$$

We want to define

$$\check{H}^i(X, \mathcal{H}) \longrightarrow \check{H}^{i+1}(X, \mathcal{F}).$$

Cheating a little, we may assume that we have a commutative diagram with exact rows,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^i(\mathcal{U}, \mathcal{F}) & \longrightarrow & C^i(\mathcal{U}, \mathcal{G}) & \longrightarrow & C^i(\mathcal{U}, \mathcal{H}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^{i+1}(\mathcal{U}, \mathcal{F}) & \longrightarrow & C^{i+1}(\mathcal{U}, \mathcal{G}) & \longrightarrow & C^{i+1}(\mathcal{U}, \mathcal{H}) & \longrightarrow & 0. \end{array}$$

Suppose we start with an element $t \in \check{H}^i(X, \mathcal{H})$. Then t is the image of $t' \in \check{H}^i(\mathcal{U}, \mathcal{H})$, for some open cover $\mathcal{U}$. In turn t' is represented by $\tau \in Z^i(\mathcal{U}, \mathcal{H})$. Now we may suppose our cover is sufficiently fine, so that $\tau_I \in \Gamma(U_I, \mathcal{H})$ is the image of $\sigma_I \in \Gamma(U_I, \mathcal{G})$ (and this fixes the cheat). Applying the boundary map, we get $\delta(\sigma) \in C^{i+1}(\mathcal{U}, \mathcal{G})$. Now the image of $\delta(\sigma)$ in $C^{i+1}(\mathcal{U}, \mathcal{H})$ is the same as $\delta(\tau)$, which is zero, as τ is a cocycle. But then by exactness of the bottom rows, we get $\rho \in C^{i+1}(\mathcal{U}, \mathcal{F})$. It is straightforward to check that ρ is a cocycle, so that we get an element $r' \in \check{H}^{i+1}(\mathcal{U}, \mathcal{F})$, whence an element r of $\check{H}^{i+1}(X, \mathcal{F})$, and that r does not depend on the choice of σ.

One can check that Čech Cohomology coincides with sheaf cohomology. In the case of a scheme, we already know that it suffices to work with any cover $\mathcal{U}$ such that U_I is affine. From now on, we won't bother to distinguish between sheaf cohomology and Čech Cohomology.