

## 12. HIGHER VANISHING

**Theorem 12.1.** *Let  $X$  be a noetherian topological space of dimension  $n$ .*

*Then*

$$H^i(X, \mathcal{F}) = 0,$$

*for all  $i > n$  and any sheaf of abelian groups.*

The basic idea is to reduce to the case of the quotient of  $\mathbb{Z}_U$ . The first thing is to reduce to the finitely generated case. Recall that any ring is the direct limit of its finitely generated subrings, so really all we need is a couple of standard results about direct limits.

Suppose  $A$  is a directed set and  $(\mathcal{F}_\alpha)$  is a direct system of sheaves indexed by  $A$ . Then we may take the direct limit  $\varinjlim \mathcal{F}_\alpha$ .

**Lemma 12.2.** *On a noetherian topological space, the direct limit of flasque is flasque.*

*Proof.* Suppose  $(\mathcal{F}_\alpha)$  is a direct system of flasque sheaves. Suppose that  $V \subset U$  are open subsets. For each  $i$  we have a surjection

$$\mathcal{F}_\alpha(U) \longrightarrow \mathcal{F}_\alpha(V).$$

Now  $\varinjlim$  is an exact functor, so

$$\varinjlim \mathcal{F}_\alpha(U) \longrightarrow \varinjlim \mathcal{F}_\alpha(V),$$

is surjective. But on a noetherian topological space we have

$$(\varinjlim \mathcal{F}_\alpha)(U) = \varinjlim \mathcal{F}_\alpha(U),$$

and so

$$(\varinjlim \mathcal{F}_\alpha)(U) \longrightarrow (\varinjlim \mathcal{F}_\alpha)(V),$$

is surjective, so that  $\varinjlim \mathcal{F}_\alpha$  is flasque. □

**Proposition 12.3.** *Let  $X$  be a noetherian topological space and let  $(\mathcal{F}_\alpha)$  be a direct system of abelian sheaves indexed by  $A$ .*

*Then there are natural isomorphisms*

$$\varinjlim H^i(X, \mathcal{F}_\alpha) \longrightarrow H^i(X, \varinjlim \mathcal{F}_\alpha)$$

*Proof.* By definition of the limit, for each  $\alpha$ , there are maps  $\mathcal{F}_\alpha \longrightarrow \varinjlim \mathcal{F}_\alpha$ . This gives a map on cohomology

$$H^i(X, \mathcal{F}_\alpha) \longrightarrow H^i(X, \varinjlim \mathcal{F}_\alpha).$$

Taking the limit of these maps gives a morphism

$$\varinjlim H^i(X, \mathcal{F}_\alpha) \longrightarrow H^i(X, \varinjlim \mathcal{F}_\alpha).$$

It is easy to check that this is an isomorphism for  $i = 0$ .

The general case follows using the notion of a  $\delta$ -functor; see (III.2.9).  $\square$

**Lemma 12.4.** *Let  $Y \subset X$  be a closed subset of a topological space, let  $\mathcal{F}$  be a sheaf of abelian groups on  $Y$  and let  $j: Y \hookrightarrow X$  be the natural inclusion.*

*Then*

$$H^i(Y, \mathcal{F}) \simeq H^i(X, j_*\mathcal{F}).$$

*Proof.* Suppose that  $\mathcal{I}^\bullet$  is a flasque resolution of  $\mathcal{F}$ . Then  $j_*\mathcal{I}^\bullet$  is a flasque resolution of  $j_*\mathcal{F}$ .  $\square$

It is customary to abuse notation and consider  $\mathcal{F}$  as a sheaf on  $X$ , without bothering to write  $j_*\mathcal{F}$ .

*Proof of (12.1).* We introduce some convenient notation. Let  $\mathcal{F}$  be a sheaf on  $X$ . If  $Y \subset X$  is a closed subset then  $\mathcal{F}_Y$  denotes the extension by zero of the sheaf  $\mathcal{F}|_Y$ ; similarly if  $U \subset X$  is an open subset, then  $\mathcal{F}_U$  denotes the extension by zero of the sheaf  $\mathcal{F}|_U$ . Note that if  $U = X - Y$  then there is an exact sequence

$$0 \longrightarrow \mathcal{F}_U \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_Y \longrightarrow 0.$$

The proof proceeds by Noetherian induction and induction on  $n = \dim X$ .

**Step 1:** We reduce to the case  $X$  is irreducible. We proceed by induction on the number  $m$  of irreducible components of  $X$ . If  $m = 1$  there is nothing to prove. Otherwise let  $Y$  be an irreducible subset and let  $U = X - Y$ . Let  $Z$  be the closure of  $U$ . Then  $\mathcal{F}_U$  can be considered as a sheaf on  $Z$ , which has  $m - 1$  irreducible components. By induction on  $m$ ,

$$H^i(Y, \mathcal{F}|_Y) = H^i(Z, \mathcal{F}|_U) = 0,$$

for all  $i > n$ , and so

$$H^i(X, \mathcal{F}) = 0,$$

for all  $i > n$ , by considering the long exact sequence of cohomology associated to the short exact sequence

$$0 \longrightarrow \mathcal{F}_U \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_Y \longrightarrow 0.$$

So we may assume that  $X$  is irreducible.

**Step 2:** Suppose that  $n = 0$ . Then  $X$  and the empty set are the only open subsets of  $X$ . In this case, to give a sheaf on  $X$  is the same as to give an abelian group, and it clear that taking global sections is an exact functor. But then

$$H^i(X, \mathcal{F}) = 0,$$

for all  $i > 0$ .

Thus we may assume that  $n > 0$ .

**Step 3:** Let

$$B = \bigcup_U \mathcal{F}(U),$$

and let  $A$  be the set of all finite subsets of  $B$ . Given  $\alpha \in A$ , let  $\mathcal{F}_\alpha$  be the subsheaf of  $\mathcal{F}$  generated by the elements of  $\alpha$ . Then  $A$  is a directed set and  $\mathcal{F} = \varinjlim \mathcal{F}_\alpha$ . By (12.3) we may therefore assume that  $\mathcal{F}$  is finitely generated.

**Step 4:** Suppose that  $\beta \subset \alpha$  and let  $r$  be the cardinality of the difference. Then there is an exact sequence

$$0 \longrightarrow \mathcal{F}_\beta \longrightarrow \mathcal{F}_\alpha \longrightarrow \mathcal{G} \longrightarrow 0,$$

where  $\mathcal{G}$  is generated by  $r$  elements. So by induction on  $r$  and the long exact sequence of cohomology associated to the short exact sequence above, we are reduced to the case when  $\mathcal{F}$  is generated by a single element, so that  $\mathcal{F}$  is a quotient of  $\mathbb{Z}_U$  for some open subset  $U$ ,

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathbb{Z}_U \longrightarrow \mathcal{F} \longrightarrow 0.$$

**Step 5:** We reduce to the case when  $\mathcal{F} = \mathbb{Z}_U$ .

For each  $x \in U$ ,  $\mathcal{R}_x \subset \mathbb{Z}_x = \mathbb{Z}$ . If  $\mathcal{R} = 0$  there is nothing to prove; otherwise let  $d$  be the smallest positive integer which appears in  $\mathcal{R}_x$ . Then

$$\mathcal{R}|_V = d \cdot \mathbb{Z}|_V,$$

for some non-empty open subset  $V \subset U$ . In this case, we have an exact sequence

$$0 \longrightarrow \mathbb{Z}_V \longrightarrow \mathcal{R} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

Now  $\mathcal{Q}$  is supported on a smaller set and so

$$H^i(X, \mathcal{Q}) = 0,$$

for all  $i \geq n$ . Taking the long exact sequence of cohomology, we see that

$$H^i(X, \mathcal{R}) = H^i(X, \mathbb{Z}_V),$$

for all  $i > n$ .

Thus we may assume that  $\mathcal{F} = \mathbb{Z}_U$ .

**Step 6:** Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z}_U \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_Y \longrightarrow 0,$$

where  $Y = X - U$ . By induction on the dimension,

$$H^i(X, \mathbb{Z}_Y) = 0,$$

for all  $i \geq n$ . Thus

$$H^i(X, \mathbb{Z}_U) = H^i(X, \mathbb{Z}),$$

for all  $i > n$ . But  $\mathbb{Z}$  is a locally constant sheaf on an irreducible space, so that  $\mathbb{Z}$  is flasque, and flasque is acyclic.  $\square$