10. The canonical bundle and divisor

Definition 10.1. *Let X be a smooth variety of dimension n over a field k. The* ***canonical sheaf****, denoted ω_X, is the highest wedge of the sheaf of relative differentials,*

$$\omega_X = \bigwedge^n \Omega_{X/k}.$$

Note that ω_X is an invertible sheaf on X. We may write $\omega_X = \mathcal{O}_X(K_X)$, for some Cartier divisor K_X. The interesting thing is that we may generalise this:

Definition 10.2. *Let X be a normal variety over a field k. Let $U \subset X$ be the smooth locus, an open subset of X, whose complement has codimension at least two.*

The ***canonical divisor****, denoted K_X, is the Weil divisor obtained by picking a Weil divisor representing the invertible sheaf ω_U and then taking the closure.*

Note that the canonical divisor is only defined up to linear equivalence.

Definition 10.3. *Let X be a smooth projective variety over a field k. The* ***geometric genus*** *of X, denoted $p_g(X)$, is the dimension of the k-vector space $H^0(X, \omega_X)$. The m-th* ***plurigenus****, denoted $P_m(X)$, is the dimension of the k-vector space $H^0(X, \mathcal{O}_X(mK_X))$. The* ***irregularity*** *of X, denoted $q(X)$, is the dimension of the k-vector space $H^0(X, \Omega_{X/k})$.*

Note that $p_g(X) = P_1(X)$. If X is a curve, then $p_g(X) = P_1(X) = q(X)$.

Theorem 10.4. *Let X and X' be two smooth projective varieties over a field k.*

If X and X' are birational then $p_g(X) = p_g(X')$, $P_n(X) = P_n(X')$ and $q(X) = q(X')$.

Proof. We will just prove that the geometric genus is a birational invariant. By symmetry, it suffices to show that $p_g(X') \leq p_g(X)$. By assumption there is a birational map $\phi X \dashrightarrow X'$. Let $V \subset X$ be the largest open subset of X for which this map restricts to a morphism, $f \colon V \longrightarrow X'$. This induces a map of sheaves,

$$f^*\Omega_{X'/k} \longrightarrow \Omega_{V/k}.$$

Since these are both locally free of the same rank $n = \dim V$, taking the highest wedge, we get

$$f^*\omega_X' \longrightarrow \omega_V.$$

Since f is birational there is an open subset $U \subset V$ such that $f(U)$ is open in X' and f induces an isomorphism $U \longrightarrow f(U)$. Since a non-zero section of an invertible sheaf cannot vanish on any non-empty open subset, we have an injection on global sections

$$H^0(X', \omega_{X'}) \longrightarrow H^0(V, \omega_V).$$

So it suffices to show that the natural restriction map

$$H^0(X, \omega_X) \longrightarrow H^0(V, \omega_V),$$

is an isomorphism.

First off, we note that the codimension of the complement $X - V$ is at least two. Indeed, let P be a codimension one point. Then $\mathcal{O}_{X,P}$ is a DVR, as X is smooth. We already have a map of the generic point of X to X'. As X' is projective it is proper, so that there is a unique morphism $\operatorname{Spec} \mathcal{O}_{X,P} \longrightarrow X'$ compatible with ϕ. This morphism extends to a neighbourhood of P, so that f is defined in a neighbourhood of P, that is $P \in V$.

To show that the restriction map is bijective, it suffices to show that if $U \subset X$ is an open subset for which $\omega_X|_U \simeq \mathcal{O}_U$, we have

$$H^0(U, \mathcal{O}_U) \longrightarrow H^0(U \cap V, \mathcal{O}_{U \cap V}).$$

But this follows as $U - V$ has codimension at least two and X is normal; any function on X which is regular in codimension two is regular. □

Definition 10.5. *Let Y be a smooth subvariety of a smooth variety X over a field k, with ideal sheaf $\mathcal{I}$. The locally free sheaf $\mathcal{I}/\mathcal{I}^2$ is called the **conormal sheaf**. Its dual*

$$\mathcal{N}_{Y/X} = \mathbf{Hom}_{\mathcal{O}_Y}(\frac{\mathcal{I}}{\mathcal{I}^2}, \mathcal{O}_Y),$$

*is called the **normal sheaf** of Y in X.*

Note that by taking duals of the usual exact sequence on Y we get

$$0 \longrightarrow \mathcal{T}_Y \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{N}_{Y/X} \longrightarrow 0.$$

Theorem 10.6 (Adjunction formula)**.** *Let Y be a smooth subvariety of codimension r of a smooth variety X over a field k. Then*

$$\omega_Y \simeq \omega_X \otimes \bigwedge^r \mathcal{N}_{Y/X}.$$

If $r = 1$ then if we consider Y as a divisor on X and put $\mathcal{L} = \mathcal{O}_X(Y)$, we get

$$\omega_Y \simeq \omega_X \otimes \mathcal{L} \otimes \mathcal{O}_Y.$$

In terms of divisors,

$$K_Y = (K_X + Y)|_Y.$$

Proof. Follows from the exact sequence above, after taking highest wedge and then the dual. ☐

It is interesting to calculate the canonical divisor in the case of a smooth toric variety. To calculate the canonical divisor, we need to write down a rational (or meromorphic in the case of $\mathbb{C}$) differential form. Note that if $z_1, z_2, \dots, z_n$ are coordinates on the torus then

$$\frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge \cdots \wedge \frac{dz_n}{z_n},$$

is invariant under the action of the torus, so that the associated divisor is supported on the invariant divisor.

To calculate the zeroes and poles of this meromorphic differential, we may work locally about any invariant divisor. So we may assume that $X = U_\sigma$ is affine, isomorphic to $\mathbb{A}^1 \times \mathbb{G}_m^{n-1}$. As usual, we reduce to the case when $n = 1$, in which case we have

$$\frac{\mathrm{d}z}{z},$$

which has a simple pole at 0.

Thus this rational form has a simple pole along every invaraint divisor, that is

$$K_X + D \sim 0,$$

where D is a sum of the invariant divisors. For example,

$$-K_{\mathbb{P}^n} = H_0 + H_1 + \cdots + H_n \sim (n+1)H.$$

One can check this with the formula one gets using the Euler sequence.