1. Affine toric varieties

First some stuff about algebraic groups:

Definition 1.1. *Let G be a group. We say that G is an* ***algebraic group*** *if G is a quasi-projective variety and the two maps $m\colon G\times G \longrightarrow G$ and $i\colon G \longrightarrow G$, where m is multiplication and i is the inverse map, are both morphisms.*

It is easy to give examples of algebraic groups. Consider the group $G = \mathrm{GL}_n(K)$. In this case G is an open subset of $\mathbb{A}^{n^2}$, the complement of the zero locus of the determinant, which expands to a polynomial. Matrix multiplication is obviously a morphism, and the inverse map is a morphism by Cramer's rule. Note that there are then many obvious algebraic subgroups; the orthogonal groups, special linear group and so on. Clearly $\mathrm{PGL}_n(K)$ is also an algebraic group; indeed the quotient of an algebraic group by a closed normal subgroup is an algebraic group. All of these are affine algebraic groups.

Definition 1.2. *Let G be an algebraic group. If G is affine then we say that G is a* ***linear algebraic group****. If G is projective and connected then we say that G is an* ***abelian variety****.*

Note that any finite group is an algebraic group (both affine and projective). It turns out that any affine group is always a subgroup of a matrix group, so that the notation makes sense.

Definition 1.3. *The group $\mathbb{G}_m$ is $\mathrm{GL}_1(K)$. The group $\mathbb{G}_a$ is the subgroup of $\mathrm{GL}_2(K)$ of upper triangular matrices with ones on the diagonal.*

Note that as a group $\mathbb{G}_m$ is the set of units in K under multiplication and $\mathbb{G}_a$ is equal to K under addition, and that both groups are affine of dimension 1; in fact they are the only linear algebraic groups of dimension one, up to isomorphism.

Note that if we are given a linear algebraic group G, we get a Hopf algebra A. Indeed if A is the coordinate ring of G, then A is a K-algebra and there are maps

$$A \longrightarrow A \otimes A \qquad \text{and} \qquad A \longrightarrow A,$$

induced by the multiplication and inverse map for G (if you don't know what a Hopf algebra is, you can unwind the definitions and take this as the definition of a Hopf algebra).

It is not hard to see that the product of two algebraic groups is an algebraic group.

Definition 1.4. *The algebraic group $\mathbb{G}_m^k$ is called a* ***torus****.*

So a torus in algebraic geometry is just a product of copies of $\mathbb{G}_m$.
In fact one can define what it means to be a group scheme:

Definition 1.5. *Let $\pi\colon X \longrightarrow S$ be a morphism of schemes. We say that X is a* ***group scheme*** *over S, if there are three morphisms, $e\colon S \longrightarrow X$, $\mu\colon X \underset{S}{\times} X \longrightarrow X$ and $i\colon X \longrightarrow X$ over S which satisfy the obvious axioms.*

We can define a group scheme $\mathbb{G}_{m,\operatorname{Spec}\mathbb{Z}}$ over $\operatorname{Spec}\mathbb{Z}$, by taking

$$\operatorname{Spec}\mathbb{Z}[x, x^{-1}].$$

Given any scheme S, this gives us a group scheme $\mathbb{G}_{m,S}$ over S, by taking the fibre square. In the case when $S = \operatorname{Spec} k$, k an algebraically closed field, then $\mathbb{G}_{m,\operatorname{Spec} k}$ is $t(\mathbb{G}_m)$, the scheme associated to the quasi-projective variety $\mathbb{G}_m$. We will be somewhat sloppy and not be too careful to distinguish the two cases.

Similarly we can take

$$\mathbb{G}_{a,\operatorname{Spec}\mathbb{Z}} = \operatorname{Spec}\mathbb{Z}[x].$$

Definition 1.6. *Let G be an algebraic group and let X be a variety acted on by G, $\pi\colon G \times X \longrightarrow X$. We say that the action is* ***algebraic*** *if π is a morphism.*

For example the natural action of $\mathrm{PGL}_n(K)$ on $\mathbb{P}^n$ is algebraic, and all the natural actions of an algebraic group on itself are algebraic.

Definition 1.7. *We say that a quasi-projective variety X is a* ***toric variety*** *if X is irreducible and normal and there is a dense open subset U isomorphic to a torus, such that the natural action of U on itself extends to an action on the whole of X.*

For example, any torus is a toric variety. $\mathbb{A}^n_k$ is a toric variety. The natural torus is the complement of the coordinate hyperplanes and the natural action is as follows

$$((t_1, t_2, \dots, t_n), (a_1, a_2, \dots, a_n)) \longrightarrow (t_1a_1, t_2a_2, \dots, t_na_n).$$

More generally, $\mathbb{P}^n$ is a toric variety. The action is just the natural action induced from the action above. A product of toric varieties is toric.

One thing to keep track of are the closures of the orbits. For the torus there is one orbit. For affine space and projective space the closure of the orbits are the coordinate subspaces.

Definition 1.8. *Let M be a lattice and let $N = \operatorname{Hom}(M, \mathbb{Z})$ be the dual lattice.*

A ***strongly convex rational polyhedral cone*** *$\sigma \subset N_{\mathbb{R}} = N \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ is*

- *a* ***cone****, that is, if $v \in \sigma$ and $\lambda \in \mathbb{R}$, $\lambda \geq 0$ then $\lambda v \in \sigma$;*
- ***polyhedral****, that is, σ is the intersection of finitely many half spaces;*
- ***rational****, that is, the half spaces are defined by equations with rational coefficients;*
- ***strongly convex****, that is, σ contains no linear spaces other than the origin.*

One can reformulate some of the parts of the definition of a strongly rational polyhedral cone. For example, σ is a polyhedral cone if and only if σ is the intersection of finitely many half spaces which are defined by homogeneous linear polynomials. σ is a strongly convex polyhedral cone if and only if σ is a cone over finitely many vectors which lie in a common half space (in other words a strongly convex polyhedral cone is the same as a cone over a polytope). And so on.

We first give the recipe of how to go from a fan to an affine toric variety. Suppose we start with σ. Form the dual cone

$$\check{\sigma} = \{\, u \in M_{\mathbb{R}} \mid \langle u, v \rangle \geq 0, v \in \sigma \,\}.$$

Now take the integral points,

$$S_\sigma = \check{\sigma} \cap M.$$

Then form the (semi)group algebra,

$$A_\sigma = K[S_\sigma].$$

Finally form the affine variety,

$$U_\sigma = \operatorname{Spec} A_\sigma.$$

Given a semigroup S, to form the semigroup algebra $K[S]$, start with a K-vector space with basis χ^u, as u ranges over the elements of S. Given u and $v \in S$ define the product

$$\chi^u \cdot \chi^v = \chi^{u+v},$$

and extend this linearly to the whole of $K[S]$.

Note that $K[S]$ is an integral domain so that U_σ is irreducible.

Example 1.9. *For example, suppose we start with $M = \mathbb{Z}^2$, σ the cone spanned by $(1, 0)$ and $(0, 1)$, inside $N_{\mathbb{R}} = \mathbb{R}^2$. Then $\check{\sigma}$ is spanned by the same vectors in $M_{\mathbb{R}}$. Therefore $S_\sigma = \mathbb{N}^2$, the group algebra is*

$\mathbb{C}[x, y]$ *and so we get* $\mathbb{A}^2$*. Similarly if we start with the cone spanned by* $e_1, e_2, \dots, e_n$ *inside* $N_{\mathbb{R}} = \mathbb{R}^n$ *then we get* $\mathbb{A}^n$*.*

Now suppose we start with $\sigma = \{0\}$ *in* $\mathbb{R}$*. Then* $\check{\sigma}$ *is the whole of* $M_{\mathbb{R}}$*,* S_σ *is the whole of* $M = \mathbb{Z}$ *and so* $\mathbb{C}[M] = \mathbb{C}[x, x^{-1}]$*. Taking* Spec *we get the torus* $\mathbb{G}_m$*.*

More generally we always get a torus of dimension n if we take the origin in $\mathbb{R}^n$. Note that if $\tau \subset \sigma$ is a face then $\check{\sigma} \subset \check{\tau}$ is also a face so that $U_\tau \subset U_\sigma$ is an open subset. In fact

Lemma 1.10. *Let* $\tau \subset \sigma \subset N_{\mathbb{R}}$ *be a face of the cone* σ*.*

Then we may find $u \in S_\sigma$ *such that*

(1) $\tau = \sigma \cap u^\perp$*,*
(2)
$$S_\tau = S_\sigma + \mathbb{Z}^+(-u),$$
(3) A_τ *is a localisation of* A_σ*, and*
(4) U_τ *is a principal open subset of* U_σ*.*

Proof. The fact that every face of a cone is cut out by a hyperplane is a standard fact in convex geometry and this is (1). For (2) note that the RHS is contained in the LHS by definition of a cone. If $w \in S_\tau$ then $w + p \cdot u$ is in $\check{\sigma}$ for any p sufficiently large. If we take p to be also an integer this shows that w belongs to the RHS.

Let χ^u be the monomial corresponding to u. (2) implies that A_τ is the localisation of A_σ along χ^u. This is (3) and (4) is immediate from (3). □

Since the cone $\{0\}$ is a face of every cone, the affine scheme associated to a cone always contains a torus, which is then dense.

Definition 1.11. *Let* $S \subset T$ *be a subsemigroup of the semigroup* T*. We say that* S *is* ***saturated*** *in* T *if whenever* $u \in T$ *and* $p \cdot u \in S$ *for some positive integer* p*, then* $u \in S$*.*

Given a subsemigroup $S \subset M$ saturation is always with respect to M.

Lemma 1.12. *Suppose that* $S \subset M$*.*

Then $K[S]$ *is integrally closed if and only if* S *is saturated.*

Proof. Suppose that $K[S]$ is integrally closed.

Pick $u \in M$ such that $v = p \cdot u \in S$ for some positive integer p. Let $b = \chi^u$ and $a = \chi^v \in K[S]$. Then
$$b^p = \chi^{pu} = \chi^v = a,$$

so that b is a root of the monic polynomial $x^p - a \in K[S][x]$. As we are assuming that $K[S]$ is integrally closed this implies that $b \in K[S]$ which implies that $u \in S$. Thus S is saturated.

Now suppose that S is saturated. As $K[S] \subset K[M]$ and the latter is integrally closed, the integral closure L of $K[S]$ sits in the middle, $K[S] \subset L \subset K[M]$. The torus acts naturally on $K[M]$ and this action fixes $K[S]$, so that it also fixes L. L is therefore a direct sum of eigenspaces, which are all one dimensional (a set of commuting diagonalisable matrices are simultaneously diagonalisable) that is L has a basis of the form χ^u, as u ranges over some subset of M. It suffices to prove that $u \in S$.

Since $b = \chi^u$ is integral over $K[S]$, we may find $k_1, k_2, \dots, k_p \in K[S]$ such that

$$b^p + k_1 b^{p-1} + \dots + k_p = 0.$$

We may assume that every term is in the same eigenspace as b^p. We may also assume that $k_p \neq 0$. As b^p and $k_p \neq 0$ belong to the same eigenspace, which is one dimensional, we get $b^p \in K[S]$. Thus $pu \in S$ and so $u \in S$ as S is saturated. Thus $b \in K[S]$ and $K[S]$ is integrally closed. $\square$

Note that S_σ is automatically saturated, as $\check{\sigma}$ is a rational polyhedral cone. In particular U_σ is normal.

Example 1.13. *Suppose that we start with the semigroup S generated by 2 and 3 inside $M = \mathbb{Z}$. Then*

$$K[S] = K[t^2, t^3] = K[x, y]/\langle y^2 - x^3 \rangle.$$

Note that this does come with an action of $\mathbb{G}_m$;

$$(t, x, y) \longrightarrow (t^2 x, t^3 y).$$

However the curve $V(y^2 - x^3) \subset \mathbb{A}^2$ is not normal.

In fact some authorities drop the requirement that a toric variety is normal.

An action of the torus corresponds to a map of algebras

$$A_\sigma \longrightarrow A_\sigma \underset{K}{\otimes} A_0,$$

which is naturally the restriction of

$$A_0 \longrightarrow A_0 \underset{K}{\otimes} A_0.$$

It is straightforward to check that the restricted map does land in $A_\sigma \underset{K}{\otimes} A_0$.

Lemma 1.14 (Gordan's Lemma)**.** *Let $\sigma \subset \mathbb{N}_{\mathbb{R}}$ be a strongly convex rational polyhedral cone.*

Then S_σ is a finitely generated semigroup.

Proof. Pick vectors $v_1, v_2, \ldots, v_n \in S_\sigma$ which generate the cone $\check{\sigma}$. Consider the set

$$K = \{\, v \in M \mid v = \sum t_i v_i, t_i \in [0,1] \,\}.$$

Then K is compact. As M is discrete $K \cap M$ is finite. I claim that the elements of $K \cap M$ generate the semigroup S_σ. Pick $u \in S_\sigma$. Since $u \in \check{\sigma}$ and $v_1, v_2, \ldots, v_n$ generate the cone, we may write

$$u = \sum \lambda_i v_i,$$

where $\lambda_i \in \mathbb{Q}$. Let $n_i = \llcorner \lambda_i \lrcorner$. Then

$$u - \sum n_i v_i = \sum (\lambda_i - \llcorner \lambda_i \lrcorner) v_i \in K \cap M.$$

As $v_1, v_2, \ldots, v_n \in K \cap M$ the result follows. $\square$

Gordan's lemma (1.14) implies that U_σ is of finite type over K. So U_σ is an affine toric variety.

Example 1.15. *Suppose we start with the cone spanned by e_2 and $2e_1 - e_2$. The dual cone is the cone spanned by f_1 and $f_1 + 2f_2$. Generators for the semigroup are f_1, $f_1 + f_2$ and $f_1 + 2f_2$. The corresponding group algebra is $A_\sigma = K[x, xy, xy^2]$. Suppose we call $u = x$, $v = xy$ and $w = xy^2$. Then $v^2 = x^2y^2 = x(xy^2) = uw$. Therefore the corresponding affine toric variety is given as the zero locus of $v^2 - uw$ in $\mathbb{A}^3$.*