HWK #8, DUE WEDNESDAY 05/02

1. Chapter III, 5.5, 5.6, 5.8, 5.9.
2. Let X be a scheme (or indeed any ringed space with enough injectives). Suppose that

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

is an exact sequence of $\mathcal{O}_X$-modules. Let e be the image of the global section $1 \in H^0(X, \mathcal{O}_X)$ inside $H^1(X, \mathcal{F})$, under the map

$$H^0(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{F}),$$

which comes from the long exact sequence of cohomology. Show that this sequence is split if and only if $e = 0$.
3. Let k be a field. Prove the following theorem of Grothendieck: Every locally free sheaf $\mathcal{E}$ on $\mathbb{P}^1_k$ splits as a direct sum of invertible sheaves,

$$\mathcal{E} \simeq \bigoplus_{i=1}^{r} \mathcal{O}_X(a_i),$$

for some integers $a_1, a_2, \ldots, a_k$. (*Hint: use induction on the rank r and question 2.*) Show that the integers $a_1, a_2, \ldots, a_k$ are unique up to re-ordering.