HWK #3, DUE WEDNESDAY 03/07

1. Hartshorne (II.6.4).
2. Hartshorne (II.6.5).
3. Hartshorne (II.6.5) parts (b) and (c), when $r = 2$ and $r = 3$, but now using the theory of toric varieties.
4. Let X be a variety. We say that a Weil divisor D is **$\mathbb{Q}$-Cartier** if there is a non-zero integer m such that the Weil divisor mD is Cartier. We say that X is **$\mathbb{Q}$-factorial** if every Weil divisor is $\mathbb{Q}$-Cartier.
Let $\sigma \subset N_{\mathbb{R}}$ be a cone. We say that σ is **simplicial** if the primitive generators $v_1, v_2, \dots, v_k$ of the one dimensional faces of σ are independent vectors in the vector space $N_{\mathbb{R}}$.
Let $X = X(F)$ be the toric variety associated to a fan F. Show that X is $\mathbb{Q}$-factorial if and only if every every cone $\sigma \in F$ is simplicial.
5. Hartshorne (II.6.8).
6. Let C be a smooth projective curve. The **degree** of a Cartier divisor $D = \sum n_i P_i$ on C is the integer $\sum n_i$. If L is a line bundle on C (aka invertible sheaf) the **degree** of L is the degree of any divisor D such that $L = \mathcal{O}_C(D)$.
Let X be a variety and let $C \subset X$ be a projective curve. Let $C' \longrightarrow C$ be the normalisation of C, so that C' is a smooth projective curve. Let $i \colon C' \longrightarrow X$ be the induced morphism. If L is a line bundle define the **intersection number** of L with C, denoted $L \cdot C$, as the degree of the line bundle i^*L on C'.
If D is a Cartier divisor define the **intersection number** of D with C, denoted $D \cdot C$, as the intersection number of the line bundle $\mathcal{O}_X(D)$ with C. We say that a Cartier divisor D (respectively line bundle L) is **nef** if the intersection number of D (respectively L) with every projective curve C in X is non-negative.
(i) Show that if we fix a projective curve C, then the natural map

$$\operatorname{Pic}(X) \longrightarrow \mathbb{Z},$$

which sends a line bundle L to the intersection number $L \cdot C$ is a group homomorphism.
(ii) Show that if $|L^{\otimes m}|$ is base point free (aka $L^{\otimes m}$ is globally generated) for some positive integer m, then L is nef.
(iii) Show that if X is a projective variety and L is an ample divisor then the intersection number of L with any curve $C \subset X$ is positive.

(iv) Show that the converse to (ii) is false. Give an example of a smooth projective variety X and a nef line bundle L such that $L^{\otimes m}$ of L is never globally generated (equivalently, $|L^{\otimes m}|$ is never base point free) for any positive integer m.