HWK #2, DUE WEDNESDAY 02/22

1. Hartshorne: Chapter II, 7.1.

2. Let $f(x) \in \mathbb{Z}[a_0, a_1, \dots, a_m][x]$ and $g(x) \in \mathbb{Z}[b_0, b_1, \dots, b_n][x]$ be two polynomials of degree m and n with coefficients $a_0, a_1, \dots, a_m$ and $b_0, b_1, \dots, b_n$. Show that there is a polynomial

$$R(f, g) \in \mathbb{Z}[a_0, a_1, \dots, a_m, b_0, b_1, \dots, b_n],$$

called the **resultant** of f and g, such that if $a_0, a_1, \dots, a_m, b_0, b_1, \dots, b_n$ belong to an algebraically closed field K then f and g have a common root if and only if $R(f, g) = 0$. (*Hint: Consider the polynomials $x^i f$, $0 \leq i \leq n-1$ and $x^j g$, $0 \leq j \leq m-1$. Show that f and g have a common root if and only if these polynomials are dependent, when considered in the vector space of polynomials of degree $m+n-1$*).

3. Let K be a field and let V be a finite dimensional vector space over K.

(i) Show that the space of linear maps $\mathrm{Hom}_K(V, V)$ is naturally an irreducible affine variety over K (this is not meant to be hard).

(ii) Show the Cayley-Hamilton theorem, that every linear map $\phi\colon V \longrightarrow V$ satisfies its own characeristic polynomial equation $\det(\phi - xI) = 0$ (*Hint: it is enough to show that there is a dense open subset of matrices which satisfy their own characteristic polynomial.*)

4. Find the toric varieties corresponding to the following fans F:

(i) Let $F = \{\{0\}, \sigma_1, \sigma_2\}$ be the fan in $N_{\mathbb{R}} = \mathbb{R}^2$, where σ_1 is the cone spanned by e_1 and σ_2 is the cone spanned by e_2.

(ii) Let F be the fan in $N_{\mathbb{R}} = \mathbb{R}^2$ given by taking all cones spanned by any subset of $\{e_1, e_2, -2e_1 - e_2\}$, excluding the cone spanned by all three (*Hint: try the quadric cone in $\mathbb{P}^3$.*)

(iii) Let F be the fan given as the faces of the cone σ spanned by four vectors v_1, v_2, v_3 and v_4 in $N_{\mathbb{R}} = \mathbb{R}^3$, where the first three vectors span the lattice N and $v_1 + v_3 = v_2 + v_4$ (in other words, find the affine toric variety U_σ).

5. Let $\sigma \subset N_{\mathbb{R}} \simeq \mathbb{R}^n$ be a strongly convex rational polyhedral cone.

(i) If σ belongs to a linear subspace of codimension k then show that U_σ is isomorphic to $U_\tau \times \mathbb{G}_m^k$, where τ is the same cone as σ but now considered to be living in a subspace W of codimension k, which is spanned by elements of N.

(ii) Show that the following are equivalent:

(1) U_σ is regular,

(2) σ is spanned by vectors $v_1, v_2, \ldots, v_k \in N$ which can be extended to vectors $v_1, v_2, \ldots, v_n \in N$ which generate the lattice N, and
(3) $U_\sigma \simeq \mathbb{A}_K^k \times \mathbb{G}_m^{n-k}$.