HWK #1, DUE WEDNESDAY 02/15

1. Suppose that σ is a cone in $N_{\mathbb{R}} = \mathbb{R}^n$ and that $u_1, u_2, \ldots, u_m$ are generators of the semigroup $S_\sigma \subset M$. Show that the affine toric variety $U_\sigma \subset \mathbb{A}^m_K$ is defined by monomial equations of the form

$$x_1^{a_1} x_2^{a_2} \ldots x_m^{a_m} = x_1^{b_1} x_2^{b_2} \ldots x_n^{b_n},$$

where

$$\sum a_i u_i = \sum b_i u_i,$$

in S_σ.

2. Let $\pi \colon X \longrightarrow B$ be a projective and surjective morphism with connected fibres of dimension n, where X and B are quasi-projective and X is irreducible. Let $f \colon X \longrightarrow Y$ be a morphism of quasi-projective varieties.

If there is a point $b_0 \in B$ such that $f(\pi^{-1}(b_0))$ is a point, then $f(\pi^{-1}(b))$ is a point for every $b \in B$. This result is known as *the rigidity lemma*. You may use the results of exercise 3.22 of Hartshorne. (*Hint: consider the morphism* $f \times \pi \colon X \longrightarrow Y \times B$).

3. Recall that an abelian variety A is a connected and projective algebraic group (you may assume that a connected algebraic group is irreducible). Show that every abelian variety is a commutative group. (*Hint: consider the morphism* $A \times A \longrightarrow A$ *given by conjugation*).

If A is a commutative algebraic group and $a \in A$ then the action of A on itself by left (or right) translation defines a morphism $\tau_a \colon A \longrightarrow A$, $\tau_a(x) = x + a$. We will refer to any such morphism as a *translation*.

4. Show that if $\pi \colon A \longrightarrow B$ is a morphism of abelian varieties then π is the composition of a translation and a group homomorphism.

5. Show that if $\pi \colon G \longrightarrow H$ is a morphism of algebraic tori then π is the composition of a translation and a group homomorphism. In particular, if $G = \mathbb{G}_m$ and $H = \mathbb{G}_m^n$ and π sends the identity to the identity then there are integers $a_1, a_2, \ldots, a_n$ such that $\pi(t) = (t^{a_1}, t^{a_2}, \ldots, t^{a_n})$. (*Hint: consider the map of group algebras (aka coordinate rings)*).

6. Let A be an abelian variety. Show that every rational map $f \colon \mathbb{P}^1 \dashrightarrow A$ is constant. You may use the fact that every morphism $\pi \colon G \longrightarrow A$ is a composition of a translation and a group homomorphism, where G is a group isomorphic to either $\mathbb{G}_a$ or $\mathbb{G}_m$.

(*Just for fun:* For those who know some of the theory of complex manifolds, note that if the underlying field is $\mathbb{C}$, then every abelian variety is a complex torus. Give another proof that f is constant in this case).