MODEL ANSWERS TO HWK #8

7.1. It suffices to check that the map is an isomorphism on stalks. Suppose that $x \in X$. By assumption there are open neighbourhoods U and V of and isomorphisms $\mathcal{L}|_U \simeq \mathcal{O}_U$, $\mathcal{M}|_V \simeq \mathcal{O}_V$. Passing to the open subset $U \cap V$ we may as well assume that $\mathcal{L} = \mathcal{M} = \mathcal{O}_X$.
Let $A = \mathcal{O}_{X,x}$. Then A is a local ring and we are given a surjective A-module homomorphism $\phi\colon A \longrightarrow A$. ϕ is given by multiplication by an element a of A. Suppose that $\phi(b) = 1$. Then $ab = 1$ and so a is a unit and ϕ is an isomorphism. Thus f is an isomorphism on stalks and f is an isomorphism.
7.2. Suppose that $m > n$. As $\dim V \leq n + 1$ it follows that t_i is a linear combination of the other sections, for some $1 \leq i \leq m$. Let $\pi\colon \mathbb{P}^m \longrightarrow \mathbb{P}^{m-1}$ be the projection map which drop the ith coordinate. The composition

$$\pi \circ \phi\colon X \longrightarrow \mathbb{P}^{m-1},$$

is the morphism given by $t_0, t_1, \dots, \hat{t_i}, \dots, t_n$. So we may assume $m = n$ by induction on $m - n$.
Suppose first that $\dim |V| = \dim V - 1 = n$. In this case both $s_1, s_2, \dots, s_n$ and $t_1, t_2, \dots, t_n$ are bases of V. So there is a unique matrix $A = (a_{ij})$ such that

$$t_i = \sum a_{ij} s_j.$$

This matrix corresponds to an isomorphism $\sigma\colon \mathbb{P}^n \longrightarrow \mathbb{P}^n$ and it is clear that $\psi = \sigma \circ \phi$.
In general the image of X is contained in linear spaces Λ_i, $i = 1$ and 2 of dimension $\dim |V| = \dim V - 1$. Pick complimentary linear subspaces Λ'_i. We have already exhibited an isomorphism $\sigma_1\colon \Lambda_1 \longrightarrow \Lambda_2$, such that $\psi = \sigma_1 \circ \phi$ and we may extend this to an isomorphism of $\sigma\colon \mathbb{P}^n \longrightarrow \mathbb{P}^n$ such that $\sigma(\Lambda'_1) = \Lambda'_2$ and $\psi = \sigma \circ \phi$.
7.3. (a) Let $\mathcal{L} = \phi^*\mathcal{O}_{\mathbb{P}^n}(1)$. As $\operatorname{Pic}(\mathbb{P}^n) = \mathbb{Z}$ it follows that $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$, for some integer d. As $\mathcal{L}$ is globally generated $d \geq 0$. If $d = 0$ then $\phi(\mathbb{P}^n)$ is a point. Otherwise $d > 0$ and $\mathcal{L}$ is ample. Suppose that $C \subset \mathbb{P}^n$ is an irreducible curve. As $\mathcal{L}$ is ample, $\mathcal{L}|_C$ is not the trivial invertible sheaf. If $x \in C$ then we may find a section $\sigma \in H^0(\mathbb{P}^n, \mathcal{L})$ which does not vanish at x. As $\mathcal{L}|_C$ is not the trivial invertible sheaf, $\sigma|_C$ must vanish somewhere. Therefore the image of C is a curve. Let $X = \phi(\mathbb{P}^n)$. If $\dim X < n$, then the fibres of $\phi\colon \mathbb{P}^n \longrightarrow X$ are positive

dimensional. But then the fibres must contain curves C (just cut by hyperplanes) which are sent to a point, a contradiction.
(b) As stated, this is obviously false. Let $\phi\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^2$ be the morphism

$$[S:T] \longrightarrow [S:S:T].$$

It is clear in this case that $d=1$. The 1-uple embedding is the identity. But then we cannot hope to project from $\mathbb{P}^1$ down to $\mathbb{P}^2$.
So let's assume that the image of ϕ is non-degenerate, that is, not contained in a hyperplane. ϕ is given by a linear system. It follows that there is an invertible sheaf $\mathcal{L}$ and a collection of sections $s_1, s_2, \dots, s_a \subset H^0(\mathbb{P}^n, \mathcal{L})$. Since $\operatorname{Pic}(\mathbb{P}^n) \simeq \mathbb{Z}$, generated by $\mathcal{O}_{\mathbb{P}^n}(1)$, it follows that $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$, up to isomorphism. Let $t_0, t_1, \dots, t_N$ be the standard basis of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ given by monomials of degree d. Then the induced morphism is the d-uple embedding $\mathbb{P}^n \longrightarrow \mathbb{P}^N$. Let

$$V \subset H^0(\mathbb{P}^n, \mathcal{L}),$$

be the subvector space spanned by $s_1, s_2, \dots, s_a$. Our assumption that ϕ is non-degenerate means that $s_1, s_2, \dots, s_a$ are a basis of V. We may extend this to a basis of $H^0(\mathbb{P}^n, \mathcal{L})$ and this defines an automorphism σ of $\mathbb{P}^N$. Projecting down to the first $a+1$ coordinates gives the morphism ϕ. Finally note that applying an automorphism of $\mathbb{P}^N$ is the same as projecting from the linear space L, which is the image under σ of the space spanned by the last $N-a-1$ coordinates and an automorphism of $\mathbb{P}^n$.
7.4. (a) If $\mathcal{L}$ is ample then $\mathcal{L}^m$ is very ample, for some positive integer m. But then there is an immersion $X \longrightarrow \mathbb{P}^n_k$ for some positive integer n and it follows that X is separated.
(b) By assumption there are two open subsets U_1 and U_2 both of which are isomorphic to $\mathbb{A}^1_k$. Let $\mathcal{L}$ be an invertible sheaf on X and let $\mathcal{L}_i$ be the restriction of $\mathcal{L}$ to U_i. As $\operatorname{Pic}(U_i) = 0$ it follows that $\mathcal{L}_i \simeq \mathcal{O}_{U_i}$. Suppose that $\{p_1, p_2\}$ are the double points of X so that

$$X - \{p_1, p_2\} = U_i - \{p_i\}.$$

The section 1 on $U_1 - \{p_1\}$ corresponds to a non-vanishing section $f(x)$ on $U_2 - \{p_2\}$. It follows that $f(x) = ax^m$, for some integer m and a non-zero scalar a. Multiplying through by automorphisms of U_2 which fix p_2 we can assume that $a = 1$. Let's call this invertible sheaf $\mathcal{L}_m(a)$. If we tensor $\mathcal{L}_m(a)$ with $\mathcal{L}_n(b)$ we get the section 1 on $U_1 - \{p_1\}$ and the section $f(x) = x^{m+n}$ on $U_2 - \{p_2\}$, so that we get the sheaf $\mathcal{L}_{m+n}(ab)$. It follows that $\operatorname{Pic}(X) = \mathbb{Z} \times K^*$.
Now let's consider if any of these line bundles are ample. By symmetry we may suppose that $m \geq 0$. Sections of $\mathcal{L}_m(a)$ correspond to pairs

$g(x)$ on U_1 and $ax^m g(x)$ on U_2, where $g(x)$ is a polynomial. There are two cases. If $m > 0$ then this section always vanishes at p_2. If $m = 0$ then this section only vanishes at p_1 if $g(x)$ has a zero at p_1, in which case the section also vanishes at p_2. Either way, $\mathcal{L}_m(a)$ does not separate points.

7.5. (a) Let $\mathcal{F}$ be a coherent sheaf. By assumption there is an integer n_0 such that $\mathcal{F} \otimes \mathcal{L}^n$ is globally generated for all $n \geq n_0$. Pick $x \in X$. Then we may find $l_1, l_2, \dots, l_k \in H^0(X, \mathcal{F} \otimes \mathcal{L}^n)$ whose images generate the stalk at x. Pick $m \in \mathcal{M}$ not vanishing at x. Then $m^n l_1$, $m^n l_2$, $\dots$, $m^n l_k$ are naturally global sections of $\mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{M}^n$ which generate the stalk at x. Hence $\mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{M}^n$ is globally generated so that $\mathcal{L} \otimes \mathcal{M}$ is ample.

(b) As $\mathcal{L}$ is ample, we may pick l so that $\mathcal{M} \otimes \mathcal{L}^l$ is globally generated. If $m > 0$ is any positive integer, then

$$\mathcal{M} \otimes \mathcal{L}^{l+m} = \mathcal{M} \otimes \mathcal{L}^l \otimes \mathcal{L}^m,$$

is ample by (a). So $\mathcal{M} \otimes \mathcal{L}^n$ is ample for any $n > l$.

(c) Since $\mathcal{O}_X$ is globally generated we may find $k > 0$ so that $\mathcal{M}^k$ is globally generated. As $\mathcal{L}$ is ample then so is $\mathcal{L}^k$. But then

$$(\mathcal{L} \otimes \mathcal{M})^k = \mathcal{L}^k \otimes \mathcal{M}^k,$$

is ample by (a). It follows that

$$\mathcal{L} \otimes \mathcal{M},$$

is ample.

(d) By assumption we may find sections $l_1, l_2, \dots, l_a \in H^0(X, \mathcal{L})$ and $m_1, m_2, \dots, m_b \in H^0(X, \mathcal{M})$ such that X_{l_i} and X_{m_j} are an open affine cover of X. Consider the sections $l_i m_j \in H^0(X, \mathcal{L} \otimes \mathcal{M})$. Note that $X_{ij} = X_{l_i} \cap X_{m_j}$ is affine. Since m_j is not zero on X_{ij}, the images

$$\frac{l_{i'} m_j}{l_i m_j} = \frac{l_{i'}}{l_i},$$

generate $H^0(X_{ij}, \mathcal{O}_X)$, since the images even generate $H^0(X_{l_i}, \mathcal{O}_X)$. It follows that the sections $l_i m_j$ define an immersion of X into $\mathbb{P}^n$ into projective space such that the pullback of $\mathcal{O}_{\mathbb{P}^n}(1)$ is $\mathcal{L} \otimes \mathcal{M}$. But then $\mathcal{L} \otimes \mathcal{M}$ is very ample.

(e) First of all we know that there is a positive integer m such that $\mathcal{L}^m$ is very ample. On the other hand, by the definition of ample, we know that there is an integer m_0 such that $\mathcal{L}^n$ is globally generated for all $n \geq m_0$. Let $n_0 = m_0 + m$. If $n \geq n_0 + m$ then

$$\mathcal{L}^n = \mathcal{L}^{n-m} \otimes \mathcal{L}^m,$$

is very ample by (d).

7.6 (a) It is shown in (5.14) that if

$$S' = \bigoplus_{d \in \mathbb{N}} H^0(X, \mathcal{O}_X(d)),$$

and S is the homogeneous coordinate ring, then $S'_d = S_d$, for d sufficiently large. On the other hand, $P_X(d) = \dim_K S_d$ for d sufficiently large.

(b) If n divides r, then $\mathcal{O}_X(rD) \simeq \mathcal{O}_X$ and the result is clear. Suppose that $H^0(X, \mathcal{O}_X(nD)) \neq 0$. Pick

$$\sigma \in H^0(X, \mathcal{O}_X(nD)),$$

which is not zero. Then

$$\tau = \sigma^{\otimes r} \in H^0(X, \mathcal{O}_X(nrD)),$$

is not zero, so that it does not vanish. Thus σ does not vanish, and so

$$\mathcal{O}_X(nD) \simeq \mathcal{O}_X.$$

It follows that r divides n and we are done.

7.7 (a) There are three ways (at least) to prove this. Firstly, we could use the fact that $\mathcal{O}_{\mathbb{P}^2}(1)$ is very ample and $\mathcal{O}_{\mathbb{P}^2}(1)$ is globally generated (for example, it is very ample) to conclude that

$$\mathcal{O}_{\mathbb{P}^2}(2) = \mathcal{O}_{\mathbb{P}^2}(1) \underset{\mathcal{O}_{\mathbb{P}^2}}{\otimes} \mathcal{O}_{\mathbb{P}^2}(1),$$

is very ample by (d).

Secondly, we could check that conics separate points and tangent vectors. If p and q are two points of $\mathbb{P}^2$, then we can certainly find a line L passing through p and not passing through q. Then $2L \in |D|$ contains p and not q, and so $|D|$ separates points.

Now suppose that z is a length two scheme, with support p. The length two scheme determines a line l containing p. Pick a conic C which contains p and is not tangent to l. Then z is not a subscheme of C and $C \in |D|$, so that $|D|$ separates tangent directions.

Thirdly, we could use the fact that $\mathbb{P}^2$ is a toric variety. Let F be the standard fan for $\mathbb{P}^2$, given by σ_1, spanned by e_1 and e_2, σ_2 spanned by e_2 and $-e_1 - e_2$ and σ_3 spanned by $-e_1 - e_2$ and e_1. Let D_3 be the z-axis, corresponding to the primitive vector $-e_1 - e_2$. Then $2D_3$ is T-Cartier and $|D| = |2D_3|$.

Consider the corresponding continuous, piecewise linear integral function ϕ_{2D_3}. σ_1 does not contain the ray spanned by $-e_1 - e_2$, so that ϕ_{2D_3} is the zero function on σ_1, which is represented by the zero vector in the dual space $M_{\mathbb{P}^2}$, so that $u(\sigma_1) = 0 \in M$. ϕ_{2D_3} takes the value 0 on e_2 and -2 on $-e_1 - e_2$, so that ϕ_{2D_3} is represented by $u(\sigma_2) = -2f_1 \in M$

on σ_2. Finally, ϕ_{2D_3} takes the value 0 on e_1 and -2 on $-e_1 - e_2$, so that ϕ_{2D_3} is represented by $u(\sigma_3) = -2f_2 \in M$ on σ_3.
It is not hard to see that ϕ_{2D_3} is convex and as $u(\sigma_2)$ and $u(\sigma_3)$ are visibly different, it follows that ϕ_{2D_3} is strictly convex, and so $2D_3$ is very ample (the condition about the semigroup is automatic if $|F|$ spans $N_{\mathbb{R}}$).
(b) One way to prove this is to check that $|V|$ separates points and tangent directions. This is somewhat tedious and involved. Alternatively we can use the fact that a sublinear system corresponds to projection. In the case at hand, we must be projecting from a point of $\mathbb{P}^5$. If the morphism to $\mathbb{P}^5$ is given by

$$[x : y : z] \longrightarrow [x^2 : y^2 : z^2 : xy : xz : yz],$$

then we are clearly projecting from the point $[0 : 0 : 0 : 1 : 1 : 1]$. Projection from a point $p \in \mathbb{P}^5$ is an immersion if and only if the point p does not lie on a secant or tangent line. As the tangent lines are limits of secant lines, it suffices to prove that p does not lie on a secant line.
Note that the Veronese gets embedded in the space of conics in the dual $\mathbb{P}^2$. Points of the Veronese correspond to conics of rank one (pure squares). The secant variety corresponds to conics or rank two (a sum of two squares).
Put coordinates $a_{i,j,k}$ on $\mathbb{P}^5$, so that $a_{i,j,k}$ is the coefficient of $x^i y^j z^k$. Then conics of rank two are given by matrices

$$\begin{pmatrix} a_{(2,0,0)} & a_{(1,1,0)} & a_{(1,0,1)} \\ a_{(1,1,0)} & a_{(0,2,0)} & a_{(0,1,1)} \\ a_{(1,0,1)} & a_{(0,1,1)} & a_{(0,0,2)} \end{pmatrix},$$

which have rank two. The matrix corresponding to p is

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

which has maximal rank. So p does not lie on the secant variety of the Veronese.
(c) There are three ways to prove that we get an embedding of $\tilde{X}$ into $\mathbb{P}^4$.
The zeroth method is to try to separate points and tangent vectors. This seems even more tedious and even more like hard work than the previous problem.
Firstly, we can proceed as above. Now we are projecting from a point p belonging to the Veronese, say the point $p = [1 : 0 : 0 : 0 : 0 : 0]$.

This map is a morphism when restricted to the Veronose, away from p. A point q in the Veronese is sent to the line connecting p to q. So, the morphism to $\mathbb{P}^4$ is not injective if and only if p lies on a line l which meets the Veronese at two other points.

We check that this cannot happen. Suppose that q is another point of the Veronese. Then p and q correspond to two pure squares. Up to choice of coordinates, these correspond to x^2 and y^2. A general point of the line connecting p to q corresponds to the conic $ax^2 + by^2$. It is easy to see that is a square if and only if $[a : b] = [1 : 0]$ or $[a : b] = [0 : 1]$. So every secant line through p only meets the Veronese at one other point.

Now suppose that l is a line tangent to the Veronese at another point q. A line is tangent to the Veronese if and only if it the line spanned by y^2 and ym, where m is some other linear form. None of these ever pass through p, so the map to $\mathbb{P}^4$ is an embedding away from p.

The graph of projection from p, on the whole of $\mathbb{P}^5$, corresponds to the blow up of p (almost by definition). The exceptional divisor gets mapped isomorphicially down to $\mathbb{P}^4$. The graph of projection from p, on the Veronese, is also given by the blow up. The exceptional divisor is a copy of $\mathbb{P}^1$, which gets embedded in the big exceptional divisor as a line in $\mathbb{P}^4$. This $\mathbb{P}^1$ then gets mapped down to a line in $\mathbb{P}^4$.

Secondly we could recognise that $\mathbb{P}^2$ blown up at one point is a toric variety. In terms of the notation above, suppose that the blow up corresponds to inserting the vector $e_1 + e_2$, so we divide σ_1 into two cones τ_1, spanned by e_1 and $e_1 + e_2$ and τ_2 spanned by $e_1 + e_2$ and e_2. The corresponding linear system is given by $2D_3 - E$, where as before D_3 corresponds to the vector $-e_1 - e_2$ and E corresponds to $e_1 + e_2$ (so that E is the exceptional divisor).

We check that $\phi_{2D_3 - E}$ is strictly convex. Computing, we have $u(\tau_1) = f_2$, $u(\tau_2) = f_1$, $u(\sigma_2) = -2f_1$ and $u(\sigma_3) = 2f_2$. It is straightforward to check that then $\phi_{2D_3 - E}$ is strictly convex.

The degree of the image is given by the number of points in the intersection of two general elements of the linear system $|D - E|$. Two conics in $\mathbb{P}^2$ intersect in four points. Blowing up one point of the intersection, namely p, the strict transform of these conics are two curves which intersect in three points.

It is clear that the strict transform in $\tilde{X}$ of the lines in $\mathbb{P}^2$ through p don't intersect. $\mathcal{O}_{\tilde{X}}(D - E)$ has degree $2 - 1 = 1$, so that the images of these curves are lines in $\mathbb{P}^4$.