MODEL ANSWERS TO HWK #7

1. (i) Call a line **standard** if it is either horizontal or vertical.
It is expedient to prove an even stronger result. We prove that if $f\colon U \longrightarrow \mathbb{C}$ is any function, where U is the complement of finitely many standard lines, which restricts to a polynomial on any standard line contained in U, then f is a polynomial. We will be somewhat sloppy and say that a standard line is contained in U if it is not one of the deleted lines (strictly speaking, only the line minus finitely many points lies in U).
Note that if $V \subset U$ is obtained from U by deleting finitely many more standard lines and $f|_V$ is a polynomial, then f is a polynomial. Indeed $f|_V$ extends to a polynomial function $g\colon U \longrightarrow \mathbb{C}$. If l is a standard line in V then $f|_l$ and $g|_l$ agree on an open subset of the line and so are equal. But then $f = g$.
Let d be the smallest positive integer such that there are uncountably many real numbers r such that the restriction of f to the vertical line $x = r$ is a polynomial of degree at most d and there are uncountably many real numbers s such that the restriction of f to the horizontal line $y = s$ is a polynomial of degree at most d.
We proceed by induction on d. Suppose that $d < 0$, so that $f(x, y)$ restricts to the zero function on infinitely many horizontal and infinitely many vertical lines. If l is any standard line contained in U then the restriction of f to l is a polynomial with infinitely many zeroes, so that f must be the zero function, which is represented by the zero polynomial.
Suppose that $d \geq 0$. Note that the change of coordinates $x \longrightarrow x - a$ does not change the property that U is the complement of finitely many standard lines, that f restricted to any standard line is a polynomial and it also does not change the value of d. So we might as well assume that the x-axis is contained in U and $f(x, 0)$ is a polynomial of degree at most d. Let $g(x, y) = f(x, y) - f(x, 0)$. Then the restriction of $g(x, y)$ to every vertical line is a polynomial in y which vanishes at the origin. Let $V \subset U$ be the set obtained by deleting the line $y = 0$. Let

$$h\colon V \longrightarrow \mathbb{C},$$

be the function $h(x, y) = g(x, y)/y$. Then V is obtained from $\mathbb{C}^2$ by deleting finitely many standard lines, $h(x, y)$ is a function which when restricted to any standard line in V is a polynomial, which has degree

at most $d-1$ on uncountably many standard lines. By induction $h(x,y)$ is a polynomial function. It follows that $f(x,y) = yh(x,y) + f(x,0)$ is a polynomial function on V, whence on U. Thus $P(\mathbb{C})$ is true.
(ii) Enumerate, $c_1, c_2, \dots$ the points of $\overline{\mathbb{Q}}$ and let $h_n(y)$ (respectively $v_n(x)$) be the monic polynomial which vanishes on the first n horizontal (respectively vertical) lines. Let

$$f(x,y) = \sum_{i=0}^{\infty} h_i(y) v_i(x).$$

It is clear that $f(x,y)$ is not a polynomial. But suppose we pick a horizontal line, given by $y = b$. Then $b = c_n$ for some n and so

$$f(x,b) = \sum_{i \leq n} h_i(b) v_i(x),$$

so that $f(x,b)$ is a polynomial (and $f(x,y)$ defines a function). By symmetry the restriction of $f(x,y)$ to any vertical line is a polynomial. So $P(\overline{\mathbb{Q}})$ fails.
(iii) Clear, from (i) and (ii) and the Lefschetz principle.
2. (i) Let $\phi \in V$, $\psi \in V$ and let $\lambda \in k$. Then

$$\phi \colon z \longrightarrow X,$$

is a morphism of schemes over k, such that the unique point of z goes to x. But then ϕ corresponds to a morphism of local rings over k,

$$f \colon \mathcal{O}_{X,x} \longrightarrow \frac{k[\epsilon]}{\langle \epsilon^2 \rangle}.$$

Similarly suppose that ψ corresponds to g. Note that the function

$$m_\lambda \colon \frac{k[\epsilon]}{\langle \epsilon^2 \rangle} \longrightarrow \frac{k[\epsilon]}{\langle \epsilon^2 \rangle} \qquad \text{given by} \qquad a + b\epsilon \longrightarrow a + \lambda b\epsilon,$$

is a morphism of local rings, which is an isomorphism if and only if $\lambda \neq 0$. Let $\lambda\phi$ be the morphism of schemes corresponding to the morphism of local rings $m_\lambda \circ f$. Similarly, define a map

$$\alpha \colon \frac{k[\epsilon_1]}{\langle \epsilon_1^2 \rangle} \bigotimes_k \frac{k[\epsilon_2]}{\langle \epsilon_2^2 \rangle} \longrightarrow \frac{k[\epsilon]}{\langle \epsilon^2 \rangle},$$

by sending both $\epsilon_1$and ϵ_2 to ϵ and extend by linearity to get a morphism of local rings. Composing with the natural map

$$(f,g) \colon \mathcal{O}_{X,x} \longrightarrow \frac{k[\epsilon_1]}{\langle \epsilon_1^2 \rangle} \bigotimes_k \frac{k[\epsilon_2]}{\langle \epsilon_2^2 \rangle},$$

we get a morphism of local rings and this defines a morphism

$$\phi + \psi \colon z \longrightarrow X.$$

This defines an operation of scalar multiplication and addition of vectors, which clearly satisfy the axioms for a vector space.
(ii) If $\phi \in T_x X$ and

$$f\colon \mathcal{O}_{X,x} \longrightarrow \frac{k[\epsilon]}{\langle \epsilon^2 \rangle},$$

is the corresponding morphism of local rings, then the kernel of f contains $\mathfrak{m}^2$. On the other hand, the inverse image of $\langle \epsilon \rangle$ is by definition contained in $\mathfrak{m}$. It follows that we get a linear map of vector spaces

$$\frac{\mathfrak{m}}{\mathfrak{m}^2} \longrightarrow k\langle \epsilon \rangle \simeq k,$$

that is, an element of the dual space

$$\left(\frac{\mathfrak{m}}{\mathfrak{m}^2} \right)^*,$$

and it is not hard to see that this assignment induces a bijection.