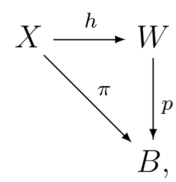
MODEL ANSWERS TO HWK #6

1. We may assume that Y is projective. Let $W \subset Y \times B$ be the closure of the image of X under the morphism $f \times \pi$. Then we may factor π into two morphisms,



where p is restriction of the second projection. Note that the second morphism is automatically projective and the first morphism is projective as the composition is projective and the second morphism is separated.

By assumption $h(\pi^{-1}(b_0))$ is a point w_0 in W. But w_0 is then the fibre of p over b_0. By upper semi-continuity of the dimensions of a fibre, it follows that there is an open subset U of B, such that $p^{-1}(b)$ is zero dimensional, for every $b \in U$. In this case, the dimension of the fibres of h over $p^{-1}(U)$ is at least n, whence the dimension of any fibre of h is at least n.

Pick $w \in W$. Then the fibre $h^{-1}(w)$ has dimension at least n. On the other hand, $h^{-1}(w) \subset \pi^{-1}(p(w))$, which has dimension n, so that $h^{-1}(w)$ is a union of some of the irreducible components of $\pi^{-1}(p(w))$. It follows that $h(\pi^{-1}(p(w))) = p^{-1}(p(w))$ is a finite set of points. As $\pi^{-1}(p(w))$ is connected, it follows that the image is a point.

2. Let $\pi\colon A \times A \longrightarrow A$ be projection onto the first factor and let $f\colon A \times A \longrightarrow A$ be the morphism which sends (g, h) to ghg^{-1}. Then $\pi^{-1}(e) = \{e\} \times A$ is sent to a point by f. As the fibres of π are irreducible of the same dimension and π is surjective, it follows that if $a \in A$ then f sends $\{a\} \times A$ to a point. As f sends (a, e) to e it follows that $aba^{-1} = e$, so that A is commutative.

3. It suffices to prove that if π sends the identity to the identity then π is a group homomomorphism. Consider the morphism of projective varieties

$$f\colon A \times A \longrightarrow B,$$

which sends (a_1, a_2) to $\pi(a_1 + a_2) - \pi(a_1) - \pi(a_2)$. Let $\phi\colon A \times A \longrightarrow A$ denote projection onto the first factor. Then f sends $\phi^{-1}(e)$ to the identity of B, where e is the identity of A. By the rigidity lemma f sends $\{a\} \times A$ to a point. But $f(a, e)$ is the identity so $f(a_1, a_2)$ is the identity of B, for every a_1 and $a_2 \in A$. But then π is a group homomomorphism.

4. We may suppose that π sends zero to zero and we need to prove that π is a group homomorphism in this case. Since $\mathbb{G}_m^n$ is a product in the category of varieties and algebraic groups, it suffices to prove this result when $H = \mathbb{G}_m$. We are given a ring homomorphism

$$K[\mathbb{Z}] \longrightarrow K[\mathbb{Z}^n],$$

which sends the maximal ideal of the origin to the maximal ideal of the origin. So we are given a semigroup homomorphism

$$\mathbb{Z} \longrightarrow \mathbb{Z}^n,$$

which sends 0 to 0. This map is determined by the image of 1. But the group homomorphism which sends x_i to t^{a_i} sends 1 to $(a_1, a_2, \dots, a_n)$. This exhausts all possibilities for where to send 1, whence the result.

5. We first show that f is a morphism. One can use the valuative criteria but it is more straightforward to prove this result directly. It suffices to prove that if we are given a rational map

$$f : \mathbb{A}^1 \longrightarrow \mathbb{P}^n,$$

then f is defined at the origin. Using the local description of morphisms, we have

$$t \longrightarrow [f_0 : f_1 : \cdots : f_n],$$

where $f_i = g_i/h_i$ is a rational function. Let $m_i = \nu(f_i)$, where ν measures the multiplicity of f_i at the origin. Let $m = \min m_i$. Then f is equally well represented by

$$t \longrightarrow [f_0' : f_1' : \cdots : f_n'],$$

where $f_i' = t^m f_i$. By our choice of m, f_i' does not have a pole at 0 and at least one f_i' is non-zero at 0. Thus f is a morphism.

We may assume that $f(0)$ is the identity of A. As $\mathbb{P}^1 - \{\infty\} \simeq \mathbb{G}_a$ it follows that $f(a+b) = f(a) + f(b)$, for all a and $b \in \mathbb{P}^1 - \{\infty\}$. As $\mathbb{P}^1 - \{0, \infty\} \simeq \mathbb{G}_m$ it follows that $f = \tau_p \circ g$, where $g(1)$ is the identity. In this case $g(ab) = g(a) + g(b)$ and so

$$f(ab) - p = g(ab) = g(a) + g(b) = f(a) + f(b) - 2p,$$

that is

$$f(ab) + p = f(a) + f(b) = f(a+b).$$

This is clearly absurd, unless $f(a)$ is the identity of A, for every $a \in \mathbb{P}^1$.

Now suppose that the groundfield is $\mathbb{C}$. Then A is a complex torus, the quotient of $\mathbb{C}^n$ by a lattice Λ of rank $2n$ and $\mathbb{P}^1$ is the Riemann sphere. The universal cover of A is $\mathbb{C}^n$ and the universal cover of $\mathbb{P}^1$ is the Riemann sphere. By the universal property of the universal cover, there is an induced commutative diagram

$$\begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{g} & \mathbb{C}^n \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \xrightarrow{f} & A. \end{array}$$

If g is not constant then one of the induced holomorphic maps

$$\mathbb{P}^1 \longrightarrow \mathbb{C},$$

given by projection, is not constant. By the open mapping theorem the image is open; as $\mathbb{P}^1$ is compact the image is compact, whence closed. The only open and closed subset of $\mathbb{C}$ is $\mathbb{C}$ itself, but this is not compact, a contradiction. Hence g is constant and so f is constant as well.

6 (i) Consider the morphism $X \times Y \longrightarrow \mathbb{G}(1, n)$. As X and Y live in complementary linear spaces this map is injective. So the image $j(X, Y)$ has dimension $d + e$. The universal family $\mathcal{J}(X, Y)$ over this has dimension $d + e + 1$ and the natural morphism to $\mathbb{P}^n$ is injective, so the image $J(X, Y)$ has dimension $d + e + 1$.

(ii) Pick Λ_1 and Λ_2 copies of $\mathbb{P}^n$ embedded as complementary linear subspaces of $\mathbb{P}^{2n+1}$. This induces $\tilde{X}$ and $\tilde{Y}$ embeddings of X and Y in $\mathbb{P}^{2n+1}$, in complementary linear spaces. By (a),

$$\dim J(\tilde{X}, \tilde{Y}) = d + e + 1.$$

Now pick a projection $\pi_\Lambda \colon \mathbb{P}^{2n+1} \dashrightarrow \mathbb{P}^n$, from a linear space Λ of dimension n, so that Λ_i get mapped isomorphically down to $\mathbb{P}^n$. For example if Λ_1 is the zero locus of $Z_{n+1}, Z_{n+2}, \dots, Z_{2n+1}$ and Λ_2 is the zero locus of $Z_0, Z_1, \dots, Z_n$ then project from the linear space $Z_i = Z_{n+1+i}$, $0 \leq i \leq n$. Consider a line $l = \langle x, y \rangle$, where $x \in \tilde{X}$ and $y \in \tilde{Y}$. Then l does not intersect Λ, since $x \neq y$ are points of $\mathbb{P}^n$, so that $J(\tilde{X}, \tilde{Y})$ does not intersect Λ.

Suppose that $z \in \mathbb{P}^n$ is a point and suppose that the fibre over z is not zero dimensional, so that $\langle \Lambda, z \rangle$ intersects $J(\tilde{X}, \tilde{Y})$ in dimension at least one. As Λ is a hyperplane in $\langle \Lambda, z \rangle$, it follows that Λ intersects $J(\tilde{X}, \tilde{Y})$, a contradiction.

But then projection down to $\mathbb{P}^n$ is morphism, with zero dimensional fibres, and so

$$\dim J(X, Y) = \dim J(\tilde{X}, \tilde{Y}) = d + e + 1.$$

(iii) If $d + e \geq n$ then $d + e + 1 > n$. By (ii) it follows that X and Y intersect.