MODEL ANSWERS TO HWK #4

1. Suppose that the point $p = [v]$ and that the plane H corresponds to $W \subset V$. Then a line l containing p, contained in H, is spanned by the vector v and a vector $w \in W$, so that, as a point of $\mathbb{P}(\bigwedge^2 V)$, $[l] = [\omega] = [v \wedge w]$. Now if W has basis v, w_1, w_2, then we can choose $w = aw_1 + bw_2$, so that the vector ω lies in the plane $v \wedge w_1$ and $v \wedge w_2$; indeed $\omega = av \wedge w_1 + bv \wedge w_2$. But this corresponds to a line L in $\mathbb{P}^5$, lying on the Grassmannian.
Now suppose that we have a line L in $\mathbb{P}^5$, lying on the Grassmannian. Any such line consists of a family $\omega = a\omega_1 + b\omega_2$ of decomposable forms, so that $\omega_i = u_i \wedge v_i$. Now if the span of the vectors u_1, u_2, v_1 and v_2 is the whole of V, then $\omega_1 + \omega_2$ is not decomposable. Otherwise v_2 is a linear combination of u_1, u_2 and v_1, so that L parametrises lines in W, the span of u_1, u_2, and v_1. But then ω_1 and ω_2 must be divisible by the same vector v (for example, by duality). Thus $p = [v]$ and $H = \mathbb{P}(W)$.
2. Suppose $p = [v]$. If the line l contains p, then it may be represented by $\omega = v \wedge w$. Suppose that we extend v to a basis v, w_1, w_2, w_3. Then we may assume that $w = a_1w_1 + a_2w_2 + a_3w_3$, so that l is represented by $a_1\omega_1 + a_2\omega_2 + a_3\omega_3$, where $\omega_i = v \wedge w_i$. Σ_p is the corresponding plane.
Now suppose that $H = \mathbb{P}(W)$. Pick a basis w_1, w_2, w_3 for W. Then a line l in H is represented by a form $\omega = a_1w_2 \wedge w_3 + a_2w_3 \wedge w_1 + a_3w_1 \wedge w_2$. Since any rank two from in a three dimensional space is automatically decomposable, the result follows easily. Alternatively, lines contained in H are the same as lines containing $[H]$ in the dual projective space.
Another way to proceed, in either case, is as follows. Consider the surface $P = \Sigma_H$. Pick any two points $[l]$ and $[m] \in P$. Then l and m are two lines in $\mathbb{P}^3$, which are contained in H. Then l and m must intersect and we set $p = l \cap m$. Then we get a line $L = \Sigma_p \cap \Sigma_H = \Sigma_{p,H} \subset P$, by 1, which contains the original two points $[l]$ and $[m] \in L$. It follows that through every two points of the surface P, we may find a unique line L. It follows easily that P is a plane. Similarly for Σ_p.
Now suppose that we are given a two plane P inside $\mathbb{G}(1, 3) \subset \mathbb{P}^5$. By 1, if $L \subset P$ is a line then there is a point $p \in \mathbb{P}^3$ and a plane $H \subset \mathbb{P}^3$ such that $L = \Sigma_{p,H}$. Suppose that we can find three lines $L_i = \Sigma_{p_i,H_i} \subset P$, $i = 1$, 2 and 3, which form a triangle $\triangle$, such that $\{p_1, p_2, p_3\}$ has cardinality three. Let $l_{ij} \subset \mathbb{P}^3$ be the line corresponding to the intersection point $L_i \cap L_j$. Then $l_{ij} = \langle p_i, p_j \rangle$. In particular p_1,

p_2 and p_3 are not collinear so that they span a plane $H = \langle p_1, p_2, p_3 \rangle$. If $H \neq H_i$ then $l_{ij} = H \cap H_i$, for $j \neq i$, a contradiction (l_{ij} must depend on j). Thus $H_1 = H_2 = H_3 = H$. Now let $L = \Sigma_{q,K} \subset P$ be an arbitrary line. Suppose that $K \neq H$. If m is the line corresponding to a point where L meets the triangle $\triangle$ then $m = H \cap K$. Since L meets the triangle $\triangle$ in at least two points, this is a contradiction. Thus $K = H$ and $P = \Sigma_H$.
It remains to deal with the case that there is no such triangle. Note that the map

$$f \colon \hat{P} \longrightarrow \mathbb{P}^3,$$

which assigns to the line $L \subset P$ the point $p \in \mathbb{P}^3$, where $L = \Sigma_{p,H}$, is a morphism. If this map is not constant then there is an open subset $U \subset \mathbb{P}^2 \times \mathbb{P}^2$ such that if $(L, M) \in U$ then $f(L) \neq f(M)$ and the image of f is not a finite set. In this case it is easy to find a triangle such that $\{p_1, p_2, p_3\}$ has cardinality three. But if f is constant then $P = \Sigma_H$.
3. Suppose that the two dimensional vector space corresponding to l_i is spanned by u_i and v_i. Let l be a line that meets l_1 at p and l_2 at q. As $p \in l_1$ and $q \in l_2$, l is represented by $\omega = (a_1u_1 + b_1v_1) \wedge (a_2u_2 + b_2v_2)$. Expanding, ω is a combination of $u_1 \wedge u_2$, $u_1 \wedge v_2$, $v_1 \wedge u_2$ and $v_1 \wedge v_2$. Let U be the span of these four vectors. In particular the locus of lines which meets l_1 and l_2 is certainly a subset of $\mathbb{P}(U)$. But the condition that any such form is decomposable, is equivalent to the condition that it is of the form $\omega = (a_1u_1 + b_1v_1) \wedge (a_2u_2 + b_2v_2)$. If we expand ω then we get the standard embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$ (up to change of sign).
Alternatively, it is clear, that abstractly the locus of lines meeting l_1 and l_2 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, as a line is specified by its intersection with l_1 and l_2.
If l_1 and l_2 intersect, then a line that meets both of them is ether a line that contains $p = l_1 \cap l_2$ or a line contained in the plane $H = \langle l_1, l_2 \rangle$. Thus the locus of lines is the union $\Sigma_p \cup \Sigma_H$, which we have seen is the union of two planes.
4. The point is that there is no moduli to this question, so that we are free to choose our favourite quadric. If we take $XW = YZ$, so that we have the image of $\mathbb{P}^1 \times \mathbb{P}^1$ under the morphism

$$([X_0 : X_1], [Y_0 : Y_1]) \longrightarrow [X_0Y_0 : X_1Y_0 : X_0Y_1 : X_1Y_1],$$

then the two families of lines are

$$[aS : aT : bS : bT] \qquad \text{and} \qquad [aS : bS : aT : bT],$$

where the pair $[a : b]$ parametrises the two families, and $[S : T]$ parametrises the lines themselves (for fixed $[a : b]$). Thus a general line from the first family is the span of

$$[a:0:b:0] \qquad \text{and} \qquad [0:a:0:b],$$

whilst a general line from the second family is the span of

$$[a:b:0:0] \qquad \text{and} \qquad [0:0:a:b].$$

Thus a line from the first (respectively second family) is represented by

$$\omega = (ae_1+be_3)\wedge(ae_2+be_4) \qquad \text{respectively} \qquad (ae_1+be_2)\wedge(ae_3\wedge be_4).$$

Expanding, the family of lines from the first family is given as

$$a^2(e_1\wedge e_2) + ab(e_1\wedge e_4 + e_3\wedge e_2) + b^2(e_3\wedge e_4),$$

and the second is given as

$$a^2(e_1\wedge e_3) + ab(e_1\wedge e_4 + e_2\wedge e_3) + b^2(e_2\wedge e_4).$$

Thus we get two conics lying in the two planes spanned by $e_1 \wedge e_2$, $e_1\wedge e_4+e_3\wedge e_2$ and $e_3\wedge e_4$, and $e_1\wedge e_3$, $e_1\wedge e_4+e_2\wedge e_3$ and $e_2\wedge e_4$. Clearly these planes do not intersect, so that they must be complementary, and neither of them is contained in $\mathbb{G}(1,3)$.
Now suppose that we have a plane conic $C \subset \mathbb{G}(1,3)$, where the span Λ of C, is not contained in $\mathbb{G}(1,3)$. In this case, by reasons of degree, $C = \Lambda \cap \mathbb{G}(1,3)$.
Suppose that when we take two general points of the conic the corresponding lines l and m intersect in $\mathbb{P}^3$. Pick a third line n. If there is a common point p to all three then the conic C meets the plane Σ_p in three points, so that the conic C must contain the line $\Sigma_p \cap \Lambda$, a contradiction. But then l, m and n must be coplanar (they lie in the plane spanned H by the three intersection points $m\cap n$, $l\cap n$ and $l\cap m$). In this case C contains three points of the plane Σ_H, so that it contains the line $\Lambda\cap\Sigma_H$, a contradiction.
So now we know that two general points of C correspond to two skew lines. There are two ways to finish. Here is the first. We may find three points of C which correspond to three skew lines m, n and m. Three skew lines have no moduli that is any three skew lines are projectively equivalent (proved in class), so there is an element $\phi \in \mathrm{PGL}_4(K)$ which carries these three lines to any other three. ϕ acts on $\mathbb{P}(\wedge^2 V)$, fixing $\mathbb{G}(1,3)$ and carries three points of the plane Λ to any other three points of $\mathbb{G}(1,3)$ which correspond to three skew lines. But any plane is determined by any three points which are not collinear and so we may assume that Λ is the plane coming from the quadric, as above.
Here is the second. $\mathbb{G}(1,3)$ is determined by a quadratic polynomial of maximal rank. This determines a bilinear form on $\wedge^2 V$ (up to scalars).

In particular given Λ there is a dual plane Λ', which is complementary to Λ and is also not contained in $\mathbb{G}(1,3)$. Let $C' = \Lambda' \cap \mathbb{G}(1,3)$, another smooth conic. Since Λ' is dual to Λ under the pairing determined by $\mathbb{G}(1,3)$ this says that if we pick $[u \wedge v] \in C$ and $[u' \wedge v'] \in C'$ then $u \wedge v \wedge u' \wedge v' = 0$, that is the corresponding lines l and l' are concurrent. So now we have two families of skew lines $\{l\}$ and $\{l'\}$ in $\mathbb{P}^3$, such that a pair of lines from both families are concurrent. Pairs of lines from both families are parametrised by $\mathbb{P}^1 \times \mathbb{P}^1$ and we get a morphism

$$\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3,$$

which sends pair (l, m) to $l \cap m$. This morphism has a bidegree, which must be $(1,1)$ since $\mathbb{P}^1 \times \{p\}$ and $\{q\} \times \mathbb{P}^1$ are both sent to a line. But then the image is the Segre, up to projective equivalence and C is just the family of lines of one ruling.

If Λ is contained in $\mathbb{G}(1,3)$ then $\Lambda = \Sigma_p$ or Σ_H. In the first case, a conic in Σ_p is the same as the family of lines in a quadric cone (which automatically pass through the vertex p of the cone). If $\Lambda = \Sigma_H$, then a conic $C \subset \Lambda$ is simply the family of tangent lines to a conic in H.

5. Let's warm up a little and see what happens if we start with the line m given by $Z_2 = Z_3 = 0$. Note that for each point p of this line we get a plane $\Sigma_p \subset \mathbb{G}(1,3)$. So we want a family of planes inside $\mathbb{G}(1,3)$. The natural guess is that this family is given by a hyperplane section. If we look at the hyperplane section $p_{34} = 0$ we get a cone over a quadric in $\mathbb{P}^3$. This is indeed covered by copies of $\mathbb{P}^2$. The condition that $p_{34} = 0$ means that that the term $e_3 \wedge e_4$ does not appear, which is the condition that we meet the line m.

(a) Since a conic degenerates to a union of two intersecting lines, the equation defininig this conic ought to be quadratic. Consider $\lambda Z_1^2 - \mu Z_0 Z_2$. If we let λ go to zero then we get $Z_0 Z_2 = 0$, the union of two lines. This gives the equation $p_{14}p_{34}$. On the other hand if we let μ go to zero we get the line $Z_1^2 = 0$ counted twice. This gives the equation $p_{24}^2 = 0$. So we guess the equation we want is some linear combination of $p_{14}p_{34}$ and $p_{24}^2 = 0$. Let's guess

$$p_{14}p_{34} = p_{24}^2.$$

Now an open subset of points of the conic has the form $[t^2 : t : 1 : 0]$. Thus an open subset of the points of the Grassmannian which intersect this conic has the form

$$\begin{pmatrix} t^2 & t & 1 & 0 \\ 0 & a & b & 1 \end{pmatrix}.$$

We have $p_{14} = t^2$, $p_{34} = 1$ and $p_{24} = t$. Clearly these set of points satisfy the equation $p_{14}p_{34} = p_{24}^2$. Now suppose we start with a line l

whose Plücker coordinates satisfy this equation. Let $A = (a_{ij})$ be a 2×4 matrix whose rows span the plane corresponding to l. If the last column is zero then $p_{i4} = 0$ and the equation holds automatically. Applying elementary row operations, we may assume that the last column is the vector $(0, 1)$. In this case $p_{i4} = a_{1i}$ and the first row has the form $(t^2, t, 1, 0)$ or it is equal to $(1, 0, 0, 0)$. Either way, this corresponds to a point on the conic.

(b) Recall that the ideal of the twisted cubic C is generated by the three quadrics $Q_0 = Z_0Z_3 - Z_1Z_2$, $Q_1 = Z_1^2 - Z_0Z_2$, $Q_2 = Z_2^2 - Z_1Z_3$. Now note that a line l intersects the twisted cubic if and only if the restrictions of Q_0, Q_1 and Q_2 to l spans a vector space of dimension at most two.

Indeed if the line l intersects C then $q_i = Q_i|_l$ all have a common zero and so cannot span the full space of quadratic polynomials on l, which has dimension three (and no common zeroes). Conversely if q_0, q_1 and q_2 span a vector space of dimension at most two then some linear combination $Q = \lambda_0 Q_0 + \lambda_1 Q_1 + \lambda_2 Q_2$ contains the line l. In this case l is a line of one of the rulings of Q, C is a curve of type $(2, 1)$ on $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and so l intersects C in one (or two) point(s).

Consider the open subset U of the Grassmannian where $p_{12} = 1$, that is consider matrices of the form

$$A = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}.$$

Natural coordinates on any line $l \in U$ are $X = Z_0$ and $Y = Z_1$. In fact at the point $(\lambda, \mu, \lambda a + \mu c, \lambda b + \mu d)$ of the line, we have

$$\begin{aligned} Z_0 &= \lambda = X \\ Z_1 &= \mu = Y \\ Z_2 &= \lambda a + \mu c = aX + cY \\ Z_3 &= \lambda b + \mu d = bX + dY. \end{aligned}$$

In this basis

$$\begin{aligned} q_0 &= bX^2 + (d - a)XY - cY^2 \\ q_1 &= -aX^2 - cXY + Y^2 \\ q_2 &= a^2X^2 - (2ac - b)XY + (c^2 - d)Y^2. \end{aligned}$$

It follows that the locus where are interested in is the rank two locus of the following matrix

$$\begin{pmatrix} b & d - a & -c \\ -a & -c & 1 \\ a^2 & 2ac - b & c^2 - d \end{pmatrix}.$$

If we expand this determinant then we get

$$-ad^2 + ac^2d + bcd + 2a^2d - bc^3 - 3abc + b^2 - a^3.$$

Note that $e = ad - bc$ is a determinant. Thus the term of degree four simplifies to

$$ac^2d - bc^3 = c^2(ad - bc) = c^2e.$$

Note that $a = -p_{23}/p_{12}$, $b = -p_{24}/p_{12}$, $c = p_{13}/p_{12}$, $d = p_{14}/p_{12}$, and $e = p_{34}/p_{12}$. Substituting and multiplying by p_{12}^3 gives an equation of degree three in the Plücker coordinates.