MODEL ANSWERS TO HWK #3

1. Let I be the ideal of V in $\mathbb{P}^n$. Then we may find $d \geq 0$ such that if we pick a basis $F_1, F_2, \dots, F_m$ for the dth graded piece I_d of I, then $F_1, F_2, \dots, F_m$ generate the ideal of V on each standard affine piece U_i of $\mathbb{P}^n$. Note that G acts fixes the ideal I, so that it acts on the vector space I_d. Let $\phi \colon X \dashrightarrow \mathbb{P}^m$ be the rational map which sends $[X_0 : X_1 : \dots : X_n]$ to $[F_0 : F_1 : \dots : F_m]$. The graph of this map gives the blow up of X, by our choice of $F_0, F_1, \dots, F_m$. But G acts on $\mathbb{P}^m = \mathbb{P}(I_m)$ and so the action of G lifts to the blow up. It is clear that this action lifts to the normalisation, by the universal property of the normalisation.

2. Define an action of $\mathbb{G}_a^n$ as follows:

$$(a_1, a_2, \dots, a_n, [X_0 : X_1 : \dots : X_n]) \longrightarrow [X_0 : X_1{+}a_1X_0 : X_2{+}a_2X_0 : \dots : X_n{+}a_nX_0].$$

This clearly extends the standard action of $\mathbb{G}_a^n$ on $\mathbb{A}^n$. It follows that the standard open affine $X_0 \neq 0$ is one orbit and that the hyperplane $X_0 = 0$ is blue.

3. Consider the $\mathbb{G}_a^2$-variety $\mathbb{P}^2$ defined in 2. Pick four blue points p, q, r and s, so that they belong to the same blue line. Let X be the blow up of $\mathbb{P}^2$ along these four points. Then X is a $\mathbb{G}_a^2$-variety by 1. Let L be the strict transform of the blue line and let E_p, E_q, E_r and E_s be the exceptional divisors lying over p, q, r and s. Then L is blue, the other points of E_p, E_q, E_r and E_s are red and the rest of the points form one dense orbit. Let

$$C = L \cup E_p \cup E_q \cup E_r \cup E_s.$$

Then L is attached to E_p, E_q, E_r and E_s at a single point.

If X and Y are two such $\mathbb{G}_a^2$-varieties, and D is the complement of the dense orbit on Y, then any $\mathbb{G}_a^2$-equivalence induces an isomorphism between C and D. If Y is obtained by blowing up p', q', r' and s', then it follows that

$$(\mathbb{P}^1, \{p, q, r, s\}) \qquad \text{and} \qquad (\mathbb{P}^1, \{p', q', r', s'\})$$

are isomorphic, that is, there is an isomorphism of $\mathbb{P}^1$ carrying the set $\{p, q, r, s\}$ to the set $\{p', q', r', s'\}$. But the j-invariant provides a continuous invariant, which takes values in $K - \{0, 1, \infty\}$, which distinguishes any such four points.

4. The fact that this is a $\mathbb{G}_a^2$-structure is clear. The orbit of $[0 : 0 : 1]$ is any point of the form $[\alpha : \beta : 1]$, so that one orbit is the standard

open subset $U_2 \simeq \mathbb{A}^2$. The orbit of $[0 : 1 : 0]$ is any point of the form $[\alpha : 1 : 0]$, so that one orbit is the line $Z = 0$, minus the point $[1 : 0 : 0]$, and so these points are red. The point $[1 : 0 : 0]$ is its own orbit, so that it is blue.

Note that this structure is not equivalent to the one given in 2. Indeed there is a blue line for the first structure and a red line for the second one.

5. It is clear that if X is a $\mathbb{G}_a^m$-variety and Y is a $\mathbb{G}_a^n$-variety, then the product $X \times Y$ is a $\mathbb{G}_a^{m+n}$-variety. In particular $\mathbb{P}^1 \times \mathbb{P}^1$ is a $\mathbb{G}_a^2$-variety. Note that there are four orbits; two red copies of $\mathbb{A}^1$ (lying in fibres of either projection), one blue point, the intersection of the closure of the two red curves and one dense orbit.

If we blow up the unique blue point, then we get a $\mathbb{G}_a^2$-variety by 1. The exceptional divisor is blue (it contains two blue points, corresponding to the strict transforms of the two red fibres, and so every point must be blue). As in lecture 1, or lecture 2, we may blow down the two red orbits. Equivalently, if take the $\mathbb{G}_a^2$-variety given in 2, and blow up two blue points, then we get the same surface, with the same $\mathbb{G}_a^2$-structure (notice that the exceptional divisors are red, since only one line through the point we blow up is blue).

6. If we can get from structure to the other by blowing up and down, then there must be surface S, which is obtained from both copies of $\mathbb{P}^2$ by only blowing up blue points.

For the first $\mathbb{G}_a^2$-variety, we can pick any blue point and blow it up. Call the resulting $\mathbb{G}_a^2$-variety S_1. For the second $\mathbb{G}_a^2$-variety, we can only blow up the blue point. Call the resulting $\mathbb{G}_a^2$-variety S_2.

Suppose that there are two birational morphisms $S \longrightarrow S_1$ and $S \longrightarrow S_2$ which blow up only blue points.

Claim 0.1. *S contains only one blue $\mathbb{P}^1$.*

Proof of (0.1). Consider, say, the birational morphism $S \longrightarrow S_1$. By hypothesis this is a composition of blow ups. By induction on the number of blow ups, we are reduced to considering what happens if we start with a $\mathbb{G}_a^2$-variety T, which contains one blue copy of $\mathbb{P}^1$, and we blow up a blue point.

Since there is only one blue $\mathbb{P}^1$, it follows that the exceptional divisor contains only one blue point, and the other points are red (more intrinsically, points of the exceptional divisor correspond to tangent directions and there is only one fixed tangent direction). The result follows by induction. □

It follows that the unique blue $\mathbb{P}^1$ on S is both the strict transform of the blue $\mathbb{P}^1$ on S_1 and the blue $\mathbb{P}^1$ on S_2. The only possibility is that S_1 and S_2 are equivalent $\mathbb{G}_a^2$-varieties. In this case there is an isomorphism of S_1 with S_2 which carries the blue curve to the blue curve. But this is not possible.

Perhaps the easiest way to see that there is no such isomorphism is to use toric geometry. On S_1 the blue curve is the strict transform of a line and on S_2 is the exceptional divisor. $\mathbb{P}^2$ blown up at a point corresponds to the fan F, by taking all of the two dimensional cones spanned by e_1, $e_1 + e_2$, e_2 and $-e_1 - e_2$. The strict transform of a line corresponds to the ray spanned by e_1 and the exceptional divisor to $e_1 + e_2$. There is no way to act by $\mathrm{SL}(2, \mathbb{Z})$ and carry e_1 to $e_1 + e_2$ fixing the fan F.