# MODEL ANSWERS TO HWK #2

1. Suppose that $X$ is contained in a coordinate hyperplane. If this coordinate hyperplane is defined by the equation $X_i = 0$ then $X$ is defined by the monomial $X_i$ and polynomials which don't involve $X_i$. Replacing $\mathbb{P}^n$ by this coordinate hyperplane and applying induction we may therefore assume that $X$ intersects the torus $G = \mathbb{G}_m^n \subset \mathbb{P}^n$. Acting by an element of $G$ won't change whether or not $X$ is defined by binomials, nor whether $X$ is a toric variety. So we may as well assume that the identity $e = [1 : 1 : 1 : \cdots : 1]$ of $G$ is contained in $X$.
By assumption we may find a dense open subset $H = \mathbb{G}_m^k \subset X$ isomorphic to a torus where $H \subset G$. Let $Z$ be the closure of $X$. Then $H$ is a dense open subset of $Z$ and the action of $H$ extends to $Z$. Now $G$ acts on $K[X_0, X_1, \dots, X_n]$ and $H$ acts on $K[X_0, X_1, \dots, X_n]$ by restriction. $K[X_0, X_1, \dots, X_n]$ decomposes as a direct sum of eigenspaces and these eigenspaces are direct sums of eigenspaces for the action of $G$, that is, the eigenspaces of the action of $H$ have a basis of monomials. Let $I$ be the ideal of $Z$. Then $I$ is invariant under the action of $H$ and so $I$ is generated by eigenpolynomials $F$.
Suppose that we pick two monomials $M_1$ and $M_2$ of the same degree with the same eigenvalue. Now $B = M_1 - M_2$ vanishes at $e$. Therefore it vanishes on the orbit of $e$, that is, on a dense subset of $Z$. Therefore $B$ vanishes on $Z$. But then it is clear that every eigenpolynomial $F \in I$ is a sum of binomials $B \in I$. It follows that $Z$ is defined by finitely many binomials $F_1, F_2, \dots, F_p$.
Note that $H$ is the complement of the coordinate hyperplanes. Let $H_i$ be a coordinate hyperplane. The stabiliser of $H_i$ in $G$ is a torus $G_i$ of dimension $n-1$ which sits naturally inside $H_i$. The intersection of $G_i$ with $H$ is a torus $H_i$ of codimension one in $H$. By induction on the dimension, it follows that the orbits of $H$ acting on $Z$ are the intersections of $Z$ with the orbits of $G$, that is, with the coordinate linear subspaces. Thus $X \subset Z$ is given by the non-vanishing of some mononomials $G_1, G_2, \dots, G_q$.
The last equivalence follows easily from what we have already proved.
2. Let $U$ be the free abelian monoid generated by $v_1, v_2, \dots, v_m$ (so that $U$ is abstractly isomorphic to $\mathbb{N}^m$). Define a monoid homomorphism $U \longrightarrow S_\sigma$ by sending $v_i$ to $u_i$. This is surjective and the kernel is

generated by relations of the form

$$\sum a_i v_i - \sum b_i v_i,$$

where

$$\sum a_i u_i = \sum b_i u_i.$$

The group algebra $A_\sigma$ is generated by $x_i = \chi^{u_i}$. Define a ring homomorphism

$$K[x_1, x_2, \dots, x_n] \longrightarrow A_\sigma,$$

by sending $x_i$ to $\chi^{u_i}$. Then the kernel is generated by equations of the form

$$x_1^{a_1} x_2^{a_2} \dots x_m^{a_m} = x_1^{b_1} x_2^{b_2} \dots x_m^{b_m},$$

where

$$\sum a_i u_i = \sum b_i u_i,$$

since if we quotient out by these relations then we get the vector space spanned by the monomials $\chi^u$, $u \in S_\sigma$.
3. Let $Z$ be the closure of $X$. Then $Z$ is an irreducible projective variety defined by the vanishing of binomials. We first prove the stronger statement that $Z$ is a non-normal toric variety. As $Z$ is defined by binomials, Hilbert's basis theorem implies that $Z$ is defined by finitely many binomials. If $Z$ is contained in a coordinate hyperplane $H$ then we might as well replace $\mathbb{P}^n$ by this coordinate hyperplane. So we may assume that $Z$ intersects the torus $G = \mathbb{G}_m^n \subset \mathbb{P}^n$. If we act by $G$ this won't change binomial equations, so that we might as well suppose that $Z$ contains the identity $e = [1 : 1 : 1 : \cdots : 1]$ so that the equations defining $Z$ take the form monomial equals monomial.
Let $W \subset \mathbb{A}^{n+1}_K$ be the affine variety defined by the same polynomials as $Z$. Suppose that $W$ is defined by monomial equations of the form

$$x_0^{a_1} x_1^{a_1} x_2^{a_2} \dots x_n^{a_n} = x_0^{b_0} x_1^{b_1} x_2^{b_2} \dots x_n^{b_n}.$$

Let $U$ be the free monoid with generators $v_0, v_1, \dots, v_n$ and let $S$ be the quotient monoid by the relations

$$\sum a_i v_i - \sum b_i v_i.$$

Then the coordinate ring of $W$ is isomorphic to $K[S]$. Embed $U \subset \mathbb{R}^n$. Then the vectors $\sum a_i v_i - \sum b_i v_i$ define a subspace $U_0$. If we project $U$ onto $U/U_0 = \mathbb{R}^m$ this defines an embedding $S \subset M \simeq \mathbb{Z}^m$. Let $\tau \subset M_{\mathbb{R}}$ be the cone spanned by the images $u_0, u_1, \dots, u_n$ of $v_0, v_1, \dots, v_n$. Then $\tau$ is a rational polyhedral cone. Let $\sigma = \check{\tau}$ be the dual cone. We may assume that $\tau$ spans $M_{\mathbb{R}}$ so that $\sigma$ is strongly convex. Then $\tau = \check{\sigma}$ and we have already seen that $S \subset S_\sigma$ is a non-normal affine toric variety defined by the same equations as $W$. In other words $W$ is a non-normal

affine toric variety. But then the action of $\mathbb{G}_m^m$ descends to an action on $Z$, so that $Z$ is a non-normal toric projective variety and the natural inclusion morphism $Z \longrightarrow \mathbb{P}^n$ is a toric morphism.
As in the proof of (1) it follows that the action of the torus $H \subset Z$ on $Z$ has only finitely many orbits which correspond to the finitely many coordinate linear subspaces. So $X$ is a union of orbits and it follows that $X$ is a toric variety and that the natural inclusion of $X$ into $\mathbb{P}^n$ is a toric morphism.