MODEL ANSWERS TO HWK #12

1. The versal deformation space is given by
$$y^2 + x^4 + ax^2 + bx + c,$$
where $(a, b, c) \in \mathbb{C}^3$ are coordinates on the base. The curve
$$y^2 + x^4 + ax^2 + bx + c = 0,$$
is singular if and only if
$$x^4 + ax^2 + bx + c$$
has a double root, that is, if and only if
$$x^4 + ax^2 + bx = c \qquad \text{and} \qquad 4x^3 + 2ax + b,$$
has a common root. Using resultants, we get
$$-4a^3b^2 + 16a^4c - 27b^4 + 144ab^2c - 128a^2c^2 + 256c^3 = 0,$$
a surface in $\mathbb{C}^3$. The curve
$$y^2 + x^4 + ax^2 + bx + c = 0,$$
has a cusp, if and only if
$$x^4 + ax^2 + bx + c$$
has a triple root, that is, if and only if
$$x^4 + ax^2 + bx + c = (x - \alpha)^3(x - \beta),$$
for some α and β. Comparing the degree three term, we must have
$$\beta = -3\alpha,$$
so that
$$x^4 + ax^2 + bx + c = (x - \alpha)^3(x + 3\alpha) = x^4 - 6\alpha^2x^2 + 8\alpha^3x - 3\alpha^3.$$
Thus the locus of curves with a cusp is given by
$$(a, b, c) = (6\alpha^2, 8\alpha^3, -3\alpha^3).$$
Similarly, the curve
$$y^2 + x^4 + ax^2 + bx + c = 0,$$
has a cusp, if and only if
$$x^4 + ax^2 + bx + c,$$
has two double roots, which happens if and only if $b = 0$.

2. If C has a tacnode, then in local coordinates C is given by

$$y^2 + x^4 = 0.$$

It is clear that

$$y^2 + x^4 \in \langle y^2, yx^2, x^4 \rangle.$$

Similarly, if C has a singularity of type A_{2n+1}, then C contains a zero dimensional scheme isomorphic to

$$z = \operatorname{Spec} \frac{\mathbb{C}[x, y]}{\langle y, x^n \rangle^2}.$$

3. Suppose we blow up the origin of $M = \mathbb{C}^3$,

$$\pi : N \longrightarrow M$$

If coordinates are $[S : T : U]$ on the exceptional divisor, then there are three coordinate patches on N:

- If $S \neq 0$, then $y = tx$ and $z = ux$.
- If $T \neq 0$, then $x = sy$ and $z = uy$.
- If $U \neq 0$, then $x = sz$ and $y = tz$.

If we start with $X = (f = 0) \subset \mathbb{C}^3$, and we denote the strict transform of X by Y, then we get an induced birational morphism

$$\psi : Y \longrightarrow X.$$

The exceptional locus C of ψ, which is a union of curves in Y, is given by intersecting the exceptional divisor of π, a copy of $\mathbb{P}^2$, with Y.
We first deal with the case of an A_n-singularity. Let

$$X = (x^2 + y^2 + z^{n+1} = 0) \subset \mathbb{C}^3.$$

Then the equation of the total transform of X on the first coordinate patch is given by

$$x^2 + (tx)^2 + (ux)^{n+1} = x^2(1 + t^2 + u^{n+1}x^{n-1}),$$

so that the equation of the strict transform Y is given by

$$1 + t^2 + u^{n+1}x^{n-1}.$$

If $x = 0$, then we get $t = \pm 1$, and these are smooth points. So nothing interesting is happening in this coordinate patch. Similarly if $T \neq 0$. The only interesting coordinate patch is given by $U \neq 0$. The equation of the total transform of X is

$$(sz)^2 + (tz)^2 + z^{n+1} = z^2(s^2 + t^2 + z^{n-1}),$$

so that the equation of the strict transform Y is given by

$$s^2 + t^2 + z^{n-1},$$

which is the equation of a surface with an A_{n-2}-singularity. The exceptional locus C is given by setting $z = 0$. Suppose that $n > 1$. In this case we get

$$s^2 + t^2 = 0,$$

which is the equation of a pair of lines, whose intersection point is the unique singular point of the strict transform, the point $s = t = z = 0$ in the third coordinate patch is given by $U \neq 0$. If we blow up one more time, then this pair of lines separates and the equations of the strict transforms do not pass through the singular point. If $n = 1$, notice that

$$s^2 + t^2 + 1 = 0,$$

is the equation of a smooth conic. It follows that we resolve singularities in $\ulcorner n/2 \urcorner$ steps, by which time we have introduced n exceptional curves, copies of $\mathbb{P}^1$, joined in a chain. Thus the resolution graph is given by the Dynkin diagram for A_n.

Now we consider what happens when we have a D_n-singularity. In this case we start with

$$X = (x^2 + y^2 z + z^{n-1} = 0) \subset \mathbb{C}^3,$$

where $n \geq 4$. If we blow up the origin, then the equation of the total transform of X on the first coordinate patch is given by

$$x^2 + (tx)^2 ux + (ux)^{n-1} = x^2(1 + t^2 x + u^{n-1} x^{n-3}),$$

so that the equation of the strict transform Y is given by

$$1 + t^2 x + u^{n-1} x^{n-3}.$$

Again, nothing interesting is happening in this coordinate patch.

The equation of the total transform of X on the second coordinate patch is given by

$$(sy)^2 + y^2(uy) + (uy)^{n-1} = y^2(s^2 + uy + u^{n-1} y^{n-3}),$$

so that the equation of the strict transform Y is given by

$$s^2 + uy + u^{n-1} y^{n-3}.$$

The exceptional locus C is given by $y = 0$, the equation of a (double) line. This divides into cases. If $n \neq 4$, then the only singular point is at the origin, where we have an A_1-singularity. If we blow up the isolated A_1-singularity we introduce another copy of $\mathbb{P}^1$. If $n = 4$, then these equations reduce to

$$s^2 + y(u + u^3),$$

and there are three singular points along C, of type A_1, at the three roots of $u(1 + u^2) = 0$. Blowing up these three singular points, we

get three copies of $\mathbb{P}^1$ and the resolution graph is given by the Dynkin diagram for D_4.
The equation of the total transform of X on the third coordinate patch is given by

$$(sz)^2 + (tz)^2 z + z^{n-1} = z^2(s^2 + t^2 z + z^{n-3}),$$

so that the equation of the strict transform Y is given by

$$s^2 + t^2 z + z^{n-3}.$$

If $n > 5$, then this is the equation of a singularity of type D_{n-2} at the origin. If $n = 4$, then we have a smooth surface. Suppose that $n = 5$, so that we have

$$s^2 + t^2 z + z^2.$$

Note that

$$s^2 + t^2 z + z^2 = s^2 + (z + t^2/2)^2 - t^4/4,$$

so that this is the equation of an A_3-singularity. Note that C is given by $z = 0$. If we blow up once, then we get two exceptional curves, both copies of $\mathbb{P}^1$. The strict transform of C passes through the unique singular point, which is an A_1-singularity. Blowing this up, in total we have the resolution graph is given by the Dynkin diagram for D_5.
Putting all of this together, we get the Dynkin diagram for D_n.
Now we consider what happens when we have a E_6-singularity. In this case we start with

$$X = (x^2 + y^3 + z^4 = 0) \subset \mathbb{C}^3.$$

If we blow up the origin, then the equation of the total transform of X on the first coordinate patch is given by

$$x^2 + (tx)^3 + (ux)^4 = x^2(1 + t^3 x + u^3 t x^2),$$

so that the equation of the strict transform Y is given by

$$1 + t^3 x + u^3 t x^2.$$

Nothing interesting is happening in this coordinate patch. The equation of the total transform of X on the second coordinate patch is given by

$$(sy)^2 + y^3 + (uy)^4 = y^2(s^2 + y + u^4 y^2)$$

so that the equation of the strict transform Y is given by

$$s^2 + y + u^4 y^2.$$

This is the equation of a smooth hypersurface.
The equation of the total transform of X on the third coordinate patch is given by

$$(sz)^2 + (tz)^3 + z^4 = z^2(s^2 + t^3 z + z^2),$$

so that the equation of the strict transform is given by

$$s^2 + t^3 z + z^2 = s^2 + (z + t^3/2)^2 - t^6/2,$$

which is the equation of an A_5-singularity. The exceptional locus C is the (double) line $s = 0$. If we blow up one more time, we get two more exceptional curves, both copies of $\mathbb{P}^1$, and the strict transform of C passes through the unique singular point, which is an A_3-singularity. Blowing up one more time, we get two more copies of $\mathbb{P}^1$, and again the strict transform of C passes through the unique singular point. Blowing up one more time introduces one more copy of $\mathbb{P}^1$ and the strict transform of C intersects the middle curve of the A_5-chain. This is the graph of given by the Dynkin diagram for E_6.

Now we consider what happens when we have a E_7-singularity. In this case we start with

$$X = (x^2 + y^3 + yz^3 = 0) \subset \mathbb{C}^3.$$

If we blow up the origin, then the equation of the total transform of X on the first coordinate patch is given by

$$x^2 + (tx)^3 + (tx)(ux)^3 = x^2(1 + t^3 x + u^3 t x^2),$$

so that the equation of the strict transform Y is given by

$$1 + t^3 x + u^3 t x^2.$$

Nothing interesting is happening in this coordinate patch. The equation of the total transform of X on the second coordinate patch is given by

$$(sy)^2 + y^3 + y(uy)^3 = y^2(s^2 + y + u^3 y^2)$$

so that the equation of the strict transform Y is given by

$$s^2 + y + u^3 y^2.$$

This is the equation of a smooth hypersurface.

The equation of the total transform of X on the third coordinate patch is given by

$$(sz)^2 + (tz)^3 + (tz)z^3 = z^2(s^2 + t^3 z + t z^2),$$

so that the equation of the strict transform is given by

$$s^2 + t^3 z + t z^2 = s^2 + (z + t^2/2)^2 t - t^5/2,$$

which is the equation of a D_6-singularity. The exceptional locus C is the (double) line $s = 0$. If we blow up one more time, we get a D_4-singularity and an A_1-singularity. The strict transform of C passes through the D_4-singularity. If we blow up the D_4-singularity, we get three A_1-singularities. The strict transform of C passes through one

of the A_1-singularities, not the one through which the previous exceptional divisor passes. Putting all of this together, the resolution graph is given by the Dynkin diagram for E_7.
Now we consider what happens when we have a E_8-singularity. In this case we start with

$$X = (x^2 + y^3 + z^5 = 0) \subset \mathbb{C}^3.$$

If we blow up the origin, then the equation of the total transform of X on the first coordinate patch is given by

$$x^2 + (tx)^3 + (ux)^5 = x^2(1 + t^3x + u^5x^3),$$

so that the equation of the strict transform is given by

$$1 + t^3x + u^5x^3.$$

Nothing interesting is happening in this coordinate patch. The equation of the total transform of X on the second coordinate patch is given by

$$(sy)^2 + y^3 + (uy)^5 = y^2(s^2 + y + u^5y^3)$$

so that the equation of the strict transform is given by

$$s^2 + y + u^5y^3.$$

This is the equation of a smooth hypersurface.
The equation of the total transform of X on the third coordinate patch is given by

$$(sz)^2 + (tz)^3 + z^5 = z^2(s^2 + t^3z + z^3),$$

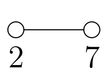
so that the equation of the strict transform is given by

$$s^2 + t^3z + z^3,$$

which is the equation of an E_7-singularity. The exceptional locus C is the (double) line $s = 0$. It is straightforward to check that the strict transform of C on the minimal resolution of the E_7-singularity, meets the curve corresponding to the vertex of degree one, which is furthest from the vertex of degree three. It follows that the resolution graph is given by the Dynkin diagram for E_8.
4. We compute the continued fraction expansion of 13/7.

$$\frac{13}{7} = 2 - \frac{1}{7}.$$

It follows that the resolution graph is a chain of 2 copies of $\mathbb{P}^1$, encoded by the following Dynkin diagram



We compute the continued fraction expansion of 15/3.

$$\begin{aligned}
\frac{15}{4} &= 4 - \frac{3}{4} \\
&= 4 - \frac{1}{\frac{4}{3}} \\
&= 4 - \frac{1}{2 - \frac{2}{3}} \\
&= 4 - \frac{1}{2 - \frac{1}{\frac{3}{2}}} \\
&= 4 - \frac{1}{2 - \frac{1}{2 - \frac{1}{2}}}.
\end{aligned}$$

It follows that the resolution graph is a chain of 4 copies of $\mathbb{P}^1$, encoded by the following Dynkin diagram

$$\underset{4}{\circ}\!-\!\!\!-\!\!\!-\underset{2}{\circ}\!-\!\!\!-\!\!\!-\underset{2}{\circ}\!-\!\!\!-\!\!\!-\underset{2}{\circ}$$

5. $X = X(F)$ is given by some fan F. A birational toric morphism $Y \longrightarrow X$ is given by repeatedly adding one dimensional rays to F and subdividing appropriately to get a fan G, so that $Y = X(G)$ is the toric variety associated to G.

Recall that if σ is a cone, then the corresponding affine toric variety U_σ is smooth if and only if the primitive vectors $v_1, v_2, \dots, v_k$ spanning the one dimensional faces of σ can be extended to a basis of the lattice N.

The first step is to reduce to the case when every cone is simplicial, that is the vectors $v_1, v_2, \dots, v_k$ are at least independent in the vector space $N_{\mathbb{R}}$. As the faces of a simplicial cone are simplicial, it suffices to reduce to the case when every maximal (with respect to inclusion) cone is simplicial. We proceed by induction on the number d of maximal cones which are not simplicial. Suppose that σ is a maximal cone which is not simplicial. Pick a vector $v \in N$ which belongs to the interior of σ. Let F' be the fan obtained from F by inserting the ray spanned by v, and subdividing accordingly. This has the result of subdividing σ into $\sigma_1, \sigma_2, \dots, \sigma_l$ simplicial subcones, and otherwise leaves every other maximal cone unchanged. It follows that F' contains one less maximal cone which is not simplicial. After d steps, we reduce to the case when every cone in F is simplicial.

Given a simplicial cone σ, let $v_1, v_2, \dots, v_k$ be the primitive generators of its one dimensional faces. Let $V \subset N_{\mathbb{R}}$ be the vector space spanned by σ (equivalently, spanned by $v_1, v_2, \dots, v_k$), and let

$$\Lambda = \mathbb{Z}v_1 + \mathbb{Z}v_2 + \cdots + \mathbb{Z}v_k,$$

be the lattice spanned by $v_1, v_2, \dots, v_k$. Then the quotient

$$\frac{N}{\Lambda},$$

is a finitely generated abelian group. Let

$$r = r_\sigma,$$

be the cardinality of the torsion part. As noted above U_σ is smooth if and only if $r_\sigma = 1$ (in fact, if σ spans N_σ, then $U_\sigma = \mathbb{C}^n/G$, for some abelian group of cardinality r_σ). Let

$$r = \max_{\sigma \in F} r_\sigma,$$

be the maximum over all cones in F. We proceed by induction on r. Pick a cone τ such that $r_\tau = r$, minimal with this property. Let $v_1, v_2, \dots, v_l$ be the primitive generators of the one dimensional faces of τ. Then we may find a vector w, in the interior of τ and belonging to the lattice N, whose image in N/Λ', where Λ' is the lattice spanned by $v_1, v_2, \dots, v_l$, is torsion. Consider the fan F' obtained by inserting the vector w. Let σ' be a cone in F' which is not in F. Then $\sigma' \subset \sigma \in F$, where σ' and σ have the same dimension. If $v_1, v_2, \dots, v_k$ are the primitive generators of the one dimensional faces of σ, then σ' has primitive generators $w, v_2, v_3, \dots, v_k$. Let Λ'' be the lattice spanned by these vectors. As the image of w in $N/Lambda$ is non-zero and torsion, it follows that the order of the torsion part of N/Λ'' is smaller than r. It follows by induction on the r and the number of cones τ such that $r_\tau = r$, that if we repeatedly insert vectors of the form w, then we eventually reduced to the case $r = 1$, in which case we have constructed a smooth toric variety Y, together with a toric birational morphism $Y \longrightarrow X$.