# MODEL ANSWERS TO HWK #11

5.5 We prove (a) and (c) by induction on the codimension of $Y$. By assumption, $Y$ is the intersection of a hypersurface of degree $d$ and another complete intersection subvariety $Z$ of codimension one less than the codimension of $Y$. There is an exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

By assumption $\mathcal{I} = \mathcal{O}_Z(-d)$. Twisting by $\mathcal{O}_Z(n)$ preserves exactness and by induction we have

$$h^i(Z, \mathcal{O}_Z(m)) = 0,$$

for all $0 < i < q + 1$ and all positive integers $m$. This gives (c) and we have

$$H^0(Z, \mathcal{O}_Z(n)) \longrightarrow H^0(Y, \mathcal{O}_Y(n)),$$

is surjective. Composing this gives (a).
(b) Note that $h^0(X, \mathcal{O}_X) = 1$ and $h^0(Y, \mathcal{O}_Y)$ is the number of connected components of $Y$. Take $n = 0$.
(d) Immediate from (c).
5.6 (a)
(1) There is an exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Q \longrightarrow 0.$$

Note that $\mathcal{I} = \mathcal{O}_X(-2)$. Twisting by $\mathcal{O}_X(n)$, we get that

$$h^1(Q, \mathcal{O}_Q(n, n)) = 0.$$

There is an exact sequence

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{O}_Q \longrightarrow \mathcal{O}_F \longrightarrow 0,$$

where $F$ is a curve of type $(1, 0)$, that is a curve of the form $\{p\} \times \mathbb{P}^1$. We have $\mathcal{J} = \mathcal{O}_Q(-1, 0)$. Twisting by $\mathcal{O}_Q(n, n)$ we have

$$0 \longrightarrow \mathcal{O}_Q(n-1, n) \longrightarrow \mathcal{O}_Q(n, n) \longrightarrow \mathcal{O}_F(n) \longrightarrow 0,$$

Now we have already seen that $h^1(Q, \mathcal{O}_Q(n, n)) = 0$ and the map

$$H^0(Q, \mathcal{O}_Q(n, n)) \longrightarrow H^0(F, \mathcal{O}_F(n)),$$

is surjective by inspection. It follows that

$$h^1(Q, \mathcal{O}_Q(n-1, n)) = 0.$$

(2) If $a \neq b$ then we might as well assume that $a < b$. Twisting the second exact sequence above by $(a+1, b)$, we get an exact sequence

$$0 \longrightarrow \mathcal{O}_Q(a,b) \longrightarrow \mathcal{O}_Q(a+1,b) \longrightarrow \mathcal{O}_F(b) \longrightarrow 0,$$

As $b < 0$, $h^0(F, \mathcal{O}_F(b)) = 0$ and so taking the long exact sequence of cohomology, we get

$$h^1(Q, \mathcal{O}_Q(a,b)) \leq h^1(Q, \mathcal{O}_Q(a+1,b)).$$

Thus we may reduce to the case $a = b$, in which case we can apply (1).
(3) Let $Y$ be a curve of type $a$, the union of $a$ copies of $\mathbb{P}^1$. Then there is an exact sequence

$$0 \longrightarrow \mathcal{O}_Q(a,0) \longrightarrow \mathcal{O}_Q \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

As

$$h^0(Y, \mathcal{O}_Y) = a \qquad \text{and} \qquad h^0(Q, \mathcal{O}_Q) = 1,$$

the result is clear.
(b)
(1) There is an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}_Q \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

As $Y$ is a curve of type $(a,b)$, we have $\mathcal{K} = \mathcal{O}_Q(-a,-b)$. (a.2) implies that

$$H^0(Q, \mathcal{O}_Q) \longrightarrow H^0(Y, \mathcal{O}_Y),$$

is surjective, so that

$$h^0(Y, \mathcal{O}_Y) \leq 1.$$

But the LHS is equal to the number of connected components of $Y$.
(2) Follows from Bertini and the fact that $Y$ is connected.
(3) By (II.5.1.4), $Y$ is projectively normal if and only if

$$H^0(X, \mathcal{O}_X(n)) \longrightarrow H^0(Y, \mathcal{O}_Y(n)),$$

is surjective for all integers $n$. Since

$$H^0(X, \mathcal{O}_X(n)) \longrightarrow H^0(Q, \mathcal{O}_Q(n)),$$

is surjective, $Y$ is projectively normal if and only if

$$H^0(Q, \mathcal{O}_Q(n)) \longrightarrow H^0(Y, \mathcal{O}_Y(n)),$$

is surjective. As $h^1(Q, \mathcal{O}_Q(n)) = 0$, this holds if and only if

$$h^1(Q, \mathcal{O}_Q(n-a, n-b)) = 0,$$

for all integers $n$.
If $|a-b| \leq 1$, then $Y$ is projectively normal by (a.1). Otherwise if $a < b-1$ and we take $n = b$, then (a.3) shows that $Y$ is not projectively normal.

(c) As always, consider the usual exact sequence

$$0 \longrightarrow \mathcal{O}_Q(n-a, n-b) \longrightarrow \mathcal{O}_Q(n) \longrightarrow \mathcal{O}_Y(n) \longrightarrow 0.$$

If $n$ is sufficiently large, then by Serre vanishing, we have that

$$\chi(Y, \mathcal{O}_Y(n)) = h^0(Y, \mathcal{O}_Y(n)) = h^0(Q, \mathcal{O}_Q(n)) - h^0(Q, \mathcal{O}_Q(n-a, n-b)).$$

Suppose $a \neq b$. Then we may suppose that $a < b$. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_Q(n-a-1, n-b) \longrightarrow \mathcal{O}_Q(n-a, n-b) \longrightarrow \mathcal{O}_F(n-b) \longrightarrow 0.$$

Assuming $n$ is sufficiently large, we get that

$$h^0(Q, \mathcal{O}_Q(n-a, n-b)) = h^0(Q, \mathcal{O}_Q(n-a-1, n-b)) + (n-b+1),$$

so that

$$h^0(Q, \mathcal{O}_Q(n-a, n-b)) = h^0(Q, \mathcal{O}_Q(n-b, n-b)) + (b-a)(n-b+1).$$

On the other hand,

$$\begin{aligned} h^0(Q, \mathcal{O}_Q(n)) &= h^0(X, \mathcal{O}_X(n)) - h^0(X, \mathcal{O}_X(n-2)) \\ &= \binom{n+3}{3} - \binom{n+1}{3} \\ &= \frac{(n+1)[(n+3)(n+2) - n(n-1)]}{6} \\ &= (n+1)^2. \end{aligned}$$

So

$$\begin{aligned} \chi(Y, \mathcal{O}_Y(n)) &= (n+1)^2 - (n+1-b)^2 - (b-a)(n-b+1) \\ &= 2b(n+1) - b^2 - bn + an + b^2 - ab - b + a \\ &= (a+b)n + a + b - ab, \end{aligned}$$

which is then the Hilbert polynomial of $Y$. It follows that $p_a(Y) = ab - a - b + 1$.

5.8 (a) Apply (II.6.7).

(b) As $\mathcal{L}$ is very ample, there is an embedding of $\tilde{X}$ into $\mathbb{P}^n$ such that $\mathcal{L} = \mathcal{O}_{\tilde{X}}(1)$. Let $H$ be a hyperplane section which avoids the inverse image of the singular locus of $X$. Then $D = H \cap \tilde{X}$ is a divisor on $\tilde{X}$ such that $D = \sum P_i$, where $Q_i = f(P_i)$ is a smooth point of $X$, for each $i$. Let $E = \sum Q_i$. Then $E$ is a Cartier divisor on $X$. Let $\mathcal{L}_0 = \mathcal{O}_X(E)$. Then $f^*\mathcal{L}_0 \simeq \mathcal{L}$. By (5.7.d), $\mathcal{L}_0$ is ample. By (II.7.6) some power of $\mathcal{L}$ is very ample and it follows by (II.5.16.1) that $X$ is projective.

(c) Let $X'$ be the disjoint union of the $X_1, X_2, \dots, X_r$. Then there is a natural morphism $i\colon X' \longrightarrow X$. This gives rise to an exact sequence

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow i_*\mathcal{O}_{X'}^* \longrightarrow \delta \longrightarrow 0,$$

where $\delta$ is supported on a zero dimensional scheme. Taking the long exact sequence of cohomology and using (4.5), we see that

$$\operatorname{Pic} X \longrightarrow \bigoplus \operatorname{Pic} X_i,$$

is surjective. Pick ample line bundles $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_r$ on $X_1, X_2, \dots, X_r$. Then we may find a line bundle $\mathcal{L}$ such that $\mathcal{L}|_{X_i} \simeq \mathcal{L}_i$. $\mathcal{L}$ is ample by (5.7.c) and so $X$ is projective.
(d) Let $\mathcal{J}$ be the ideal sheaf of $X_{\text{red}}$ in $X$. Let $\mathcal{X}_k$ be subscheme of $X$ determined by $\mathcal{I}^k$. Then $X_k \subset X_{k+1}$ and the ideal sheaf $\mathcal{I}_k$ squares to zero. By (4.6), there is an exact sequence

$$\operatorname{Pic} X_{k+1} \longrightarrow \operatorname{Pic} X_k \longrightarrow H^2(X, \mathcal{I}_k).$$

By (2.7) the last group is zero. Composing, we get that

$$\operatorname{Pic} X \longrightarrow \operatorname{Pic} X_{\text{red}},$$

is surjective. Let $\mathcal{M}$ be an ample line bundle on $X_{\text{red}}$ and let $\mathcal{L}$ be a line bundle on $X$ whose restriction to $X_{\text{red}}$ is $\mathcal{M}$. (5.7.c) implies that $\mathcal{L}$ is ample.
5.9. Note that

$$\frac{x_j}{x_i}\,\mathrm{d}\left(\frac{x_i}{x_j}\right) = \frac{1}{x_i x_j}(x_j\,\mathrm{d}x_i - x_i\,\mathrm{d}x_j) = \frac{\mathrm{d}x_i}{x_i} - \frac{\mathrm{d}x_j}{x_j}.$$

It follows that we do have a 1-cocycle.
As in the hint, we just have to show that $\delta(\mathcal{O}_X(1)) \neq 0$. Note that

$$\mathcal{L}|_{U_i} \simeq \mathcal{O}_{U_i},$$

via the map which sends $f \in \mathcal{O}_{U_i}(V)$ to $X_i f$, so that $\mathcal{O}_X(1)$ is represented by the 1-cocycle

$$\frac{x_1}{x_0} \quad \text{on } U_{01}, \qquad \frac{x_2}{x_1} \quad \text{on } U_{12}, \text{ and} \qquad \frac{x_0}{x_2} \quad \text{on } U_{20}.$$

Let $\mathcal{U} = \{U_0, U_1, U_2\}$. The relevant commutative diagram is then

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^1(\mathcal{U},\omega) & \longrightarrow & C^1(\mathcal{U},\mathcal{O}_{X'}^*) & \longrightarrow & C^1(\mathcal{U},\mathcal{O}_X^*) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^2(\mathcal{U},\omega) & \longrightarrow & C^2(\mathcal{U},\mathcal{O}_{X'}^*) & \longrightarrow & C^2(\mathcal{U},\mathcal{O}_X^*) & \longrightarrow & 0.
\end{array}$$

2. Note that $e = 0$ if and only if $1 \in H^0(X, \mathcal{O}_X)$ is in the image of

$$\phi\colon H^0(X, \mathcal{G}) \longrightarrow H^0(X, \mathcal{O}_X).$$

Suppose that $\phi(\sigma) = 1$. Define a sheaf homomorphism

$$\mathcal{O}_X \longrightarrow \mathcal{G},$$

by sending $f \in \mathcal{G}(U)$ to $f\sigma|_U$. It is easy to see that this defines a splitting.

Conversely, if the exact sequence is split, it is clear that $\phi$ is surjective.

3. Let $X = \mathbb{P}^1$. Pick $m$ such that $\mathcal{E}(m)$ is globally generated. Then we get a morphism of sheaves

$$\mathcal{O}_X \longrightarrow \mathcal{E}(m).$$

If we dualise this exact sequence we get an exact sequence

$$\mathcal{E}^*(-m) \longrightarrow \mathcal{O}_X.$$

Let $\mathcal{K}$ be the kernel. If we pick a point $p \in X$, we get an exact sequence on stalks,

$$0 \longrightarrow \mathcal{K}_p \longrightarrow \mathcal{O}_{X,p}^r \longrightarrow \mathcal{O}_{X,p} \longrightarrow 0,$$

where

$$\mathcal{O}_{X,p} \simeq k[x]_x.$$

As $k[x]_x$ is a PID, it follows that

$$\mathcal{K}_p \simeq \mathcal{O}_{X,p}^{r-1},$$

so that $\mathcal{K}$ is locally free of rank $r-1$. Denoting by $\mathcal{Q}$ the dual of $\mathcal{K}$, we get an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E}(m) \longrightarrow \mathcal{Q} \longrightarrow 0.$$

As $\mathcal{Q}$ is locally free of rank $r-1$, by induction on $r$, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E}(m) \longrightarrow \bigoplus_{i=1}^{r-1} \mathcal{O}_X(a_i) \longrightarrow 0,$$

for some integers $a_1, a_2, \dots, a_{r-1}$.

Suppose that we tensor this exact sequence by $\mathcal{O}_X(-k)$, where

$$k > \max_i a_i,$$

and $k > 0$. Then $h^0(X, \mathcal{E}(m-k)) = 0$, since both $h^0(X, \mathcal{O}_X(-k)) = 0$ and $h^0(X, \mathcal{Q}(-k)) = 0$. It follows that there is a smallest integer $m'$ such that $h^0(X, \mathcal{E}(m')) \neq 0$. Replacing $m$ by $m'$, we get the same exact sequence as before, but in addition we also have $h^0(X, \mathcal{E}(m-1)) = 0$. As

$$h^1(X, \mathcal{O}_X(-1)) = h^0(X, \omega_X(1)) = h^0(X, \mathcal{O}_X(-1)) = 0,$$

it follows that

$$h^0(X, \mathcal{Q}(-1)) \leq 1.$$

In particular $a_i - 1 \leq 0$, that is $a_i \leq 1$. It follows that

$$h^1(X, \mathcal{K}) = \sum_i h^1(X, \mathcal{O}_X(-a_i)) = \sum_i h^0(X, \mathcal{O}_X(a_i - 2)) = 0.$$

But then (2) implies that the short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{E}^*(-m) \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

is split, so that $\mathcal{E}^*(-m)$ is a direct sum of line bundles. But then $\mathcal{E}(m)$ is a direct sum of the dual line bundles and so $\mathcal{E}$ is also a direct sum of line bundles.

It is easy to see that one can recover the sequence

$$a_1, a_2, \dots, a_r,$$

from the data of the

$$h^0(X, \mathcal{E}(d)),$$

for all integers $d$.