MODEL ANSWERS TO HWK #10

4.1 Let $\mathcal{U} = \{U_i\}$ be an open affine cover of X which is locally finite. Then $\mathcal{V} = \{V_i\}$ is an open affine cover of Y which is locally finite, where $V_i = f^{-1}(U_i)$. Note that, if I is a finite set of indices, then there are natural isomorphisms

$$H^0(U_I, \mathcal{F}) = H^0(V_I, f_*\mathcal{F}).$$

It follows that

$$H^*(\mathcal{U}, \mathcal{F}) \simeq H^*(\mathcal{V}, f_*\mathcal{F}).$$

But, by (II.5.8) we have that $f_*\mathcal{F}$ is quasi-coherent and so

$$H^*(X, \mathcal{F}) \simeq H^*(\mathcal{U}, \mathcal{F}) \qquad \text{and} \qquad H^*(Y, \mathcal{F}) \simeq H^*(\mathcal{V}, f_*\mathcal{F}).$$

4.2 (a) Let M and L be the functions fields of X and Y (that is, the residue fields of the generic points of X and Y). Then M/L is a finite field extension. Pick a basis $m_1, m_2, \dots, m_r$ of the L-vector space M. By assumption $X = \operatorname{Spec} A$ and then M is the field of fractions of A. We may find $m \in A$ and $a_1, a_2, \dots, a_r \in A$ such that

$$m_i = \frac{a_i}{m}.$$

Let $\mathcal{M}$ be the invertible sheaf corresponding to the Cartier divisor given by m. Then there is a morphism of $\mathcal{O}_Y$-modules,

$$\alpha \colon \mathcal{O}_Y^r \longrightarrow f_*\mathcal{M},$$

which is an isomorphism at the generic point, since then it reduces to the vector space isomorphism,

$$L^r \longrightarrow M.$$

(b) If we apply $\mathcal{H}\text{om}(\cdot, \mathcal{F})$ to α, we get a morphism of sheaves

$$\mathcal{H}\text{om}(f_*\mathcal{M}, \mathcal{F}) \longrightarrow \mathcal{F}^r,$$

which is certainly an isomorphism at the generic point. Note that

$$\mathcal{H}\text{om}(f_*\mathcal{M}, \mathcal{F}),$$

is a coherent $\mathcal{A} = f_*\mathcal{O}_X$-module. By (5.17e), there is a coherent $\mathcal{O}_X$-module $\mathcal{G}$ such that

$$f_*\mathcal{G} = \mathcal{H}\text{om}(f_*\mathcal{M}, \mathcal{F}).$$

(c) Let $Y' \subset Y$ be a closed subscheme, let $X' \subset f^{-1}(Y')$ be a closed subset of the inverse such that the induced morphism $f' \colon X' \longrightarrow Y'$

is surjective. Note that $X' \subset X$ is affine, and f' is a finite morphism. Indeed, to check f' is finite, we may assume that $Y = \operatorname{Spec} B$ is affine and by assumption A is a finitely generated B-module. If I and J are the defining ideals of X' and Y', then $X' = \operatorname{Spec} A/I$, $Y' = \operatorname{Spec} B/J$ and it is clear that A/I is a finitely generated B/J-module.
Note that $f_{\mathrm{red}} \colon X_{\mathrm{red}} \longrightarrow Y_{\mathrm{red}}$ is a surjective finite morphism of noetherian, separated and reduced schemes. As (3.1) implies that Y is affine if and only if Y_{red} is affine, we may assume that X and Y are reduced.
Suppose that $Y' \subset Y$ is an irreducible component of Y. As f is surjective, there is an irreducible component X' of X which surjects to Y'. The induced morphism $f' \colon X' \longrightarrow Y$ is a surjective finite morphism of noetherian, separated and integral schemes. As (3.2) implies that Y is affine if and only if each irreducible component Y' is affine, we may assume that X and Y are integral. Let $\mathcal{F}$ be a quasi-coherent sheaf on Y. We check that

$$H^i(Y, \mathcal{F}) = 0,$$

for all $i > 0$. By Noetherian induction and (3.7), we may suppose that

$$H^i(Y', \mathcal{G}) = 0,$$

for all proper closed subsets and all quasi-coherent sheaves $\mathcal{G}$.
By (b), we may find an exact sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow f_*\mathcal{G} \longrightarrow \mathcal{F}^r \longrightarrow \mathcal{Q} \longrightarrow 0,$$

where $\mathcal{R}$ and $\mathcal{Q}$ are quasi-coherent sheaves, supported on proper closed subsets of Y. By induction,

$$H^i(Y, \mathcal{F}^r) = H^i(Y, f_*\mathcal{G}),$$

and the last group is isomorphic to

$$H^i(X, \mathcal{G}),$$

by (4.1). But this vanishes as X is affine and $\mathcal{G}$ is quasi-coherent. Thus

$$H^i(Y, \mathcal{F}) = 0,$$

for all $i > 0$ and all quasi-coherent sheaves $\mathcal{F}$, and so Y is affine by (3.7).
4.3 Let $\mathcal{U} = \{U_x, U_y\}$, where U_x is the complement of the x-axis and U_y is the complement of the y-axis. Then U_x and U_y are both isomorphic to $\mathbb{A}^1 \times (\mathbb{A}^1 - \{0\})$, so that they are both affine. The intersection of U_x and U_y is $(\mathbb{A}^1 - \{0\}) \times (\mathbb{A}^1 - \{0\})$, which is again affine. As $\mathcal{O}_X$ is coherent, we have an isomorphism,

$$H^1(\mathcal{U}, \mathcal{O}_X) \simeq H^1(X, \mathcal{O}_X).$$

Now an element of $C^1(\mathcal{U}, \mathcal{O}_X)$ is nothing but a section of $H^0(U_x \cap U_y, \mathcal{O}_X)$. Since there are no triple intersections, every cochain is automatically a cocycle, so that

$$Z^1(\mathcal{U}, \mathcal{O}_X) = C^1(\mathcal{U}, \mathcal{O}_X) = k[x, y]_{xy}.$$

Now

$$C^0(\mathcal{U}, \mathcal{O}_X) = H^0(U_x, \mathcal{O}_X) \oplus H^0(U_y, \mathcal{O}_X).$$

Note that

$$H^0(U_x, \mathcal{O}_X) = k[x, y]_x \qquad \text{and} \qquad H^0(U_y, \mathcal{O}_X) = k[x, y]_y.$$

Thus

$$B^1(\mathcal{U}, \mathcal{O}_X) = k[x, y]_x + k[x, y]_y.$$

It follows that a basis of

$$H^1(X, \mathcal{O}_X),$$

is given by monomials of the form $x^i y^j$, where $i < 0$ and $j < 0$. In particular,

$$h^1(X, \mathcal{O}_X),$$

is not finite.

It is also interesting to calculate $H^1(X, \mathcal{O}_X)$ using the fact that X is toric. The fan F corresponding to X is the union of the two one dimensional cones spanned by e_1 and e_2 (but not including the cone spanned by e_1 and e_2) and the origin (which is a face of both one dimensional cones). Then the support of the fan F is

$$|F| = \{\, (x, 0) \mid x \geq 0 \,\} \cup \{\, (0, y) \mid y \geq 0 \,\}.$$

The 0 divisor is T-Cartier and corresponds to the zero function on F. According to (9.10),

$$H^1(X, \mathcal{O}_X),$$

decomposes as a direct sum of eigenspaces, indexed by $u \in M$, where each piece is given by a local cohomology group,

$$H^1_{Z(u)}(|F|, \mathbb{C}).$$

The last group is isomorphic to the relative cohomology of the pair

$$H^1(|F|, Z(u), \mathbb{C}).$$

The long exact sequence for the pair $Z(u) \subset |F|$ is:

$$0 \longrightarrow H^0(|F|, |F| - Z(u), \mathbb{C}) \longrightarrow H^0(|F| - Z(u), \mathbb{C}) \longrightarrow H^0(|F|, \mathbb{C}) \dashrightarrow$$
$$\dashrightarrow H^1(|F|, |F| - Z(u), \mathbb{C}) \longrightarrow H^1(|F| - Z(u), \mathbb{C}) \longrightarrow H^1(|F|, \mathbb{C}) \longrightarrow 0.$$

Note that $H^0(|F|, \mathbb{C}) = \mathbb{C}$ and $H^1(|F| - Z(u), \mathbb{C})$ is always trivial. It follows that

$$H^1_{Z(u)}(|F|, \mathbb{C}),$$

is non-trivial, equal to $\mathbb{C}$, if and only if $|F| = Z(u)$, if and only if $u = (i, j)$, where $i \leq 0$ and $j \leq 0$.

4.5 As in the hint any invertible sheaf $\mathcal{L}$ determines an element $l_{\mathcal{U}}$ of $H^1(\mathcal{U}, \mathcal{O}^*_X)$, where $\mathcal{L}|_{U_i}$ is trivial. If $\mathcal{V}$ is a refinement of $\mathcal{U}$, then $\mathcal{L}|_{V_j}$ is certainly trivial, where $V_j \subset U_i$, and it is easy to check that

$$l_{\mathcal{V}} \in H^1(\mathcal{V}, \mathcal{O}^*_X),$$

is the same as the image of $l_{\mathcal{U}}$ under the natural map

$$H^1(\mathcal{U}, \mathcal{O}^*_X) \longrightarrow H^1(\mathcal{V}, \mathcal{O}^*_X).$$

Thus $\mathcal{L}$ determines an element of the direct limit. Using (5.4) this gives us a map

$$\pi\colon \operatorname{Pic}(X) \longrightarrow H^1(X, \mathcal{O}^*_X).$$

If $\mathcal{L}$ and $\mathcal{M}$ are two invertible sheaves, then there is a common cover $\mathcal{U}$ over which they are both trivial. It is easy to see that the image of $\mathcal{L} \underset{\mathcal{O}_X}{\otimes} \mathcal{M}$ in $H^1(\mathcal{U}, \mathcal{O}^*_X)$ is $l_{\mathcal{U}} + m_{\mathcal{U}}$. But then π is a group homomorphism. To give an element of $H^1(X, \mathcal{O}^*_X)$ is to give an element of $H^1(\mathcal{U}, \mathcal{O}^*_X)$, for some open cover $\mathcal{U}$. Using this 1-cocycle, one can construct an invertible sheaf, $\mathcal{L}$, which represents this 1-cocycle. Thus π is surjective. Suppose that $\mathcal{L}$ is sent to zero. Then there is some open cover $\mathcal{U}$ for which the corresponding 1-cocycle is a coboundary, represented by

$$\sigma_i \in H^0(U_i, \mathcal{O}^*_X).$$

But then σ defines a global non-vanishing section of $\mathcal{L}$, so that

$$\mathcal{L} \simeq \mathcal{O}_X.$$

It follows that π is injective.

4.7 TBC

5.1 We can split the long exact sequence of cohomology into one short exact sequence,

$$0 \longrightarrow H^0(X, \mathcal{F}') \longrightarrow H^0(X, \mathcal{F}) \longrightarrow Q \longrightarrow 0,$$

and one long exact sequence, which starts with

$$0 \longrightarrow Q' \longrightarrow H^1(X, \mathcal{F}') \longrightarrow H^1(X, \mathcal{F}) \dots,$$

where

$$Q' = \frac{H^0(X, \mathcal{F}'')}{Q}.$$

We have

$$h^0(X, \mathcal{F}) = h^0(X, \mathcal{F}') + \dim_k Q,$$

and, by an obvious induction,

$$\sum_{i \geq 1}(-1)^{i-1}h^i(X, F) = \sum_{i \geq 1}(-1)^{i-1}h^i(X, F') + \sum_{i \geq 0}(-1)^{i-1}h^i(X, F'') - \dim_k Q.$$

Adding the two equations together gives the result.

5.2 (a) Pick a divisor Y belonging to the linear system determined by $\mathcal{O}_X(1)$. Note that there is a morphism of sheaves

$$\mathcal{F}(-1) \longrightarrow \mathcal{F},$$

which is locally given by multiplication by the defining equation of Y, so that this map is an isomorphism away from Y. We get an exact sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{F}(-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{Q} \longrightarrow 0,$$

where $\mathcal{Q}$ and $\mathcal{R}$ are defined to fix exactness. Note that $\mathcal{Q}$ and $\mathcal{R}$ are coherent and they are both supported on Y. If we tensor this exact sequence by $\mathcal{O}_X(n)$ we get

$$0 \longrightarrow \mathcal{R}(n) \longrightarrow \mathcal{F}(n-1) \longrightarrow \mathcal{F}(n) \longrightarrow \mathcal{Q}(n) \longrightarrow 0,$$

By (5.1) we have

$$\Delta\chi(\mathcal{F}(n)) = \chi(\mathcal{Q}(n)) - \chi(\mathcal{R}(n)).$$

By Noetherian induction the RHS is a polynomial and so $\chi(\mathcal{F}(n))$ is also a polynomial.

(b) By Serre vanishing, there is an integer n_0 such that

$$\chi(\mathcal{F}(n)) = h^0(\mathbb{P}^n, \mathcal{F}(n)),$$

for $n \geq n_0$. But we have already seen that the RHS is precisely the dimension of the nth graded piece of $\Gamma_*(\mathcal{F})$.

5.3 (a) If X is integral, and k is an algebraically closed field, then there is a projective variety X' such that $t(X') = X$. We have that

$$H^0(X', \mathcal{O}_{X'}) = H^0(X, \mathcal{O}_X).$$

But by (I.3.4), the LHS is isomorphic to k.

(b) Clear from (5.2).

(c) Let $f\colon C \dashrightarrow X$ be a rational map from a smooth curve to a projective variety. Then f is a morphism. Thus if $f\colon C_1 \dashrightarrow C_2$ is a birational map, then f is in fact an isomorphism. It is then clear that $p_a(C)$ is a birational invariant.

If C is a smooth plane curve of degree d then the arithmetic genus of C is

$$\binom{d-1}{2}.$$

In particular, if $d \geq 3$, the arithmetic genus of C is non-zero, so that C is not rational.

5.7 (a) Let $\mathcal{F}$ be any coherent sheaf on Y. Then $\mathcal{G} = i_*\mathcal{F}$ is a coherent sheaf on X. As $\mathcal{L}$ is ample, there is an integer n_0 such that if $n \geq n_0$, then

$$H^i(X, \mathcal{G} \otimes \mathcal{L}^n) \qquad \text{for any} \qquad n \geq n_0, i > 0.$$

On the other hand,

$$H^i(Y, \mathcal{F} \otimes i^*\mathcal{L}^n) = H^i(X, \mathcal{G} \otimes \mathcal{L}^n).$$

(b) Since X_{red} is a closed subscheme, (a) implies that $\mathcal{L}_{\text{red}}$ is ample. Now suppose that $\mathcal{L}_{\text{red}}$ is ample. Let $\mathcal{F}$ be a quasi-coherent sheaf on X and let $\mathcal{N}$ be the sheaf of nilpotent elements. Then

$$\mathcal{N}^k \cdot \mathcal{L} = 0,$$

for some $k > 0$. Let $\mathcal{G} = \mathcal{N} \cdot \mathcal{F}$. By induction on k, there is a constant n_0 such that

$$H^i(X, \mathcal{G} \otimes \mathcal{L}^n) = 0,$$

for all $n \geq n_0$. There is a short exact sequence,

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow \mathcal{H} \longrightarrow 0,$$

where $\mathcal{H}$ is supported on X_{red}. Possibly increasing n_0, we may assume that

$$H^i(X, \mathcal{H} \otimes \mathcal{L}^n) = 0,$$

for all all $n \geq n_0$. Tensoring by $\mathcal{L}^n$ and taking the long exact sequence of cohomology, we get

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0,$$

all $n \geq n_0$. But then $\mathcal{L}$ is ample by (5.3).

(c) As X_i is a closed subscheme of X, (a) implies that $\mathcal{L} \otimes \mathcal{O}_{X_i}$ is ample. Let $\mathcal{I}$ be the ideal sheaf of X_1. Let $\mathcal{F}$ be a quasi-coherent sheaf. Then there is an exact sequence

$$0 \longrightarrow \mathcal{I} \cdot \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0,$$

where $\mathcal{G}$ is a quasi-coherent sheaf supported on X_1. Tensoring by a sufficiently high power of $\mathcal{L}$ and by induction on the number of irreducible components, taking the long exact sequence of cohomology, we get that

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0,$$

all $n \geq n_0$. But then $\mathcal{L}$ is ample by (5.3).
(d) If $\mathcal{L}$ is ample, and $\mathcal{F}$ is a quasi-coherent sheaf on X, then $f_*\mathcal{F}$ is quasi-coherent sheaf on Y and

$$H^i(X, \mathcal{F} \otimes f^*\mathcal{L}^n) = H^i(Y, f_*\mathcal{F} \otimes \mathcal{L}^n) = 0,$$

for all n sufficiently large. Hence $f^*\mathcal{L}$ is ample.
For the other direction, by (b) and (c) we may suppose that X and Y are integral. Let $\mathcal{F}$ be a quasi-coherent sheaf on Y. As in the proof of (4.2), we may find an exact sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow f_*\mathcal{G} \longrightarrow \mathcal{F}^r \longrightarrow \mathcal{Q} \longrightarrow 0,$$

where $\mathcal{R}$ and $\mathcal{Q}$ are quasi-coherent sheaves, supported on proper closed subsets of Y, and $\mathcal{G}$ is a coherent sheaf on X. Tensoring by a high power of $\mathcal{L}$, applying Noetherian induction, we get

$$H^i(Y, \mathcal{F}^r \otimes \mathcal{L}^n) = H^i(X, \mathcal{G} \otimes f^*\mathcal{L}^n) = 0,$$

for all $i > 0$. But then $\mathcal{L}$ is ample.