MODEL ANSWERS TO HWK #1

5.6 (a) See the lecture notes.
(b) Changing coordinates, and working locally, we may assume that $P = (0, 0)$ is the origin of $\mathbb{A}^2$. Let $f(x, y) \in K[x, y]$ be a defining equation. By assumption

$$f(x, y) = f_2(x, y) + f_3(x, y) + \dots,$$

where $f_i(x, y)$ is homogeneous of degree i and $f_2(x, y) = lm$, where l and m are two homogeneous polynomials of degree one, which are independent elements of the vector space of polynomials. Changing coordinates, we may assume that $l = x + y$ and $m = y$, so that

$$f(x, y) = (x + y)y + f_3(x, y) + \dots.$$

Let $\pi \colon X \longrightarrow \mathbb{A}^2$ be the blow up of $\mathbb{A}^2$. Suppose that we have coordinates $[S : T]$ on $\mathbb{P}^1$. Then X is covered by two copies of $\mathbb{A}^2$. On the locus where $S \neq 0$, we have natural coordinates x and $t = T/S$. The equation for the total transform is then

$$f(x, xt) = (x + xt)xt + f_3(x, xt) + \dots.$$

Note that $f_i(x, xt)$ is divisible by x^i, so that

$$f_i(x, xt) = x^i g(x, t),$$

for some polynomial $g(x, t) \in K[x, t]$. It follows that we may factor out x^2 from $f(x, xt)$ and so the equation for the strict transform is given by

$$t(1 + t) + xg_3(x, xt) + \dots.$$

If we set $x = 0$, then we get $t = 0$ or $t = -1$, and these are the two points where the strict transform meets the exceptional divisor, at least on the coordinate patch where $S \neq 0$.
If $T \neq 0$, then natural coordinates are y and $s = S/T$. The equation for the total transform is then

$$f(ys, y) = (ys + y)y + f_3(ys, y) + \dots.$$

As before we may write

$$f_i(ys, y) = y^i g_i(y, s),$$

and so the equation for the strict transform is then

$$(s + 1) + yg_3(y, s) + \dots.$$

Clearly this only meets the exceptional divisor at the point $s = -1$ (which is the same as the point $t = -1$).
(c) We are looking at the polynomial $y^2 + x^4$. On the coordinate patch $S \neq 0$, we have

$$y^2 + x^4 = (xt)^2 + x^4 = x^2(t^2 + x^2),$$

so that the equation for the strict transform is $t^2 + x^2 = 0$. As

$$t^2 + x^2 = (t + ix)(t - ix),$$

where $i^2 = -1$, this is the equation of a curve with a node. It is clear that we can resolve this in two steps. Note that if we compute in the other coordinate patch, where $T \neq 0$, then the strict transform is even disjoint from the exceptional divisor.
(d) On the coordinate patch $S \neq 0$, we have

$$y^3 + x^5 = (xt)^3 + x^5 = x^3(t^3 + x^2),$$

so that the equation for the strict transform is $t^3 + x^2 = 0$. This is the equation of a curve with a cusp. One more blow up resolves this singularity.
5.7 (a) Let $v \in \mathbb{A}^3$ be a point, not the origin and let $[v] \in \mathbb{P}^2$ be the corresponding point of $\mathbb{P}^2$. By assumption, one of

$$\frac{\partial f}{\partial x}\Big|_{[v]} \qquad \frac{\partial f}{\partial y}\Big|_{[v]} \qquad \text{and} \qquad \frac{\partial f}{\partial z}\Big|_{[v]},$$

does not vanish at $[v]$. But then the corresponding partial does not vanish at v. In particular, the only possible singular point is P. As the degree of f is greater than one, all of the partials of f vanish at P, so that P is the unique singular point of X.
(b) As ϕ is an isomorphism outside of P, the singular locus of $\tilde{X}$ is located along the exceptional divisor of ϕ. Note that $\tilde{X}$ is the strict transform of X inside the blow up of $\mathbb{A}^3$ at P. Introduce coordinates (x, y, z) on $\mathbb{A}^3$ and $[R : S : T]$ on $\mathbb{P}^2$. Equations for the blow up of $\mathbb{A}^3$ are given by

$$xS = yR \qquad xT = zR \qquad \text{and} \qquad yT = zS.$$

Equations for the total transform on the open set $T \neq 0$ are given by

$$f(zr, zs, z) = 0.$$

As f is homogeneous of degree d, it follows that we may write

$$f(zr, zs, z) = z^d f(r, s, 1),$$

so that $f(r, s, 1) = 0$ is the equation of the strict transform, that is, the equation for the blow up $\tilde{X}$ in the chart $\mathbb{A}^3$, where $T \neq 0$. This is clearly smooth, and by symmetry $\tilde{X}$ is smooth.

(c) The curve $f(r, s, 1) = 0$ and $z = 0$ is the equation of the exceptional divisor of ϕ, on the chart $T \neq 0$. This is the same as the equation of the curve $f(X, Y, Z) = 0$ on the chart $Z \neq 0$. So, by symmetry, $\phi^{-1}(P)$ is clearly isomorphic to the original curve.