9. Sheaf Cohomology

Definition 9.1. *Let X be a topological space. For every $i \geq 0$ there are functors H^i from the category of sheaves of abelian groups on X to the category of abelian groups such that*

(1) $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$.
(2) *Given a short exact sequence,*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0,$$

there are coboundary maps

$$H^i(X, \mathcal{H}) \longrightarrow H^{i+1}(X, \mathcal{F}).$$

which can be strung together to get a long exact sequence of cohomology.

In short, sheaf cohomology was invented to fix the lack of exactness, and in fact this property essentially fixes the definition.

Example 9.2. *If X is a simplicial complex (or a CW-complex) then $H^i(X, \mathbb{Z})$ agrees with the usual definition. The same goes for any other coefficient ring (considered as a locally constant sheaf).*

Like ordinary cohomology, sheaf cohomology inherits a cup product,

$$H^i(X, \mathcal{F}) \otimes H^j(X, \mathcal{G}) \longrightarrow H^{i+j}(X, \mathcal{F} \otimes \mathcal{G}),$$

where $(X, \mathcal{O}_X)$ is a ringed space and $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules. In particular if X is a projective scheme over A then

$$H^i(X, \mathcal{F}),$$

is an A-module, where $\mathcal{F}$ is an $\mathcal{O}_X$-module, since there is a ring homorphism $A \longrightarrow H^0(X, \mathcal{O}_X)$. In particular if A is a field, then

$$H^i(X, \mathcal{F}),$$

are vector spaces.

$$h^i(X, \mathcal{F}),$$

denotes their dimension.

We would like to have a definition of these groups which allows us to compute. Let X be a topological space and let $\mathcal{U} = \{U_i\}$ be an open cover, which is locally finite. The group of k-cochains is

$$C^k(\mathcal{U}, \mathcal{F}) = \bigoplus_I \Gamma(U_I, \mathcal{F}),$$

where I runs over all $(k+1)$-tuples of indices and

$$U_I = \bigcap_{i \in I} U_i,$$

denotes intersection. k-cochains are skew-commutative, so that if we switch two indices we get a sign change.

Define a coboundary map

$$\delta^k \colon C^k(\mathcal{U}, \mathcal{F}) \longrightarrow C^{k+1}(\mathcal{U}, \mathcal{F}).$$

Given $\sigma = (\sigma_I)$, we have to construct $\tau = \delta(\sigma) \in C^{k+1}(\mathcal{U}, \mathcal{F})$. We just need to determine the components τ_J of τ. Now $J = \{j_0, j_1, \dots, j_k\}$. If we drop an index, then we get a k-tuple. We define

$$\tau_J = \left(\sum_{i=0}^{k} (-1)^i \sigma_{J-\{i_i\}} \right) \Bigg|_{U_J}.$$

The key point is that $\delta^2 = 0$. So we can take cohomology

$$H^i(\mathcal{U}, \mathcal{F}) = Z^i(\mathcal{U}, \mathcal{F}) / B^i(\mathcal{U}, \mathcal{F}).$$

Here Z^i denotes the group of i-cocycles, those elements killed by δ^i and B^i denotes the group of coboundaries, those cochains which are in the image of δ^{i-1}. Note that $\delta^i(B^i) = \delta^i \delta^{i-1}(C^{i-1}) = 0$, so that $B^i \subset Z^i$.

The problem is that this is not enough. Perhaps our open cover is not fine enough to capture all the interesting cohomology. A **refinement** of the open cover $\mathcal{U}$ is an open cover $\mathcal{V}$, together with a map h between the indexing sets, such that if V_j is an open subset of the refinement, then for the index $i = h(j)$, we have $V_j \subset U_i$. It is straightforward to check that there are maps,

$$H^i(\mathcal{U}, \mathcal{F}) \longrightarrow H^i(\mathcal{V}, \mathcal{F}),$$

on cohomology. Taking the (direct) limit, we get the Čech cohomology groups,

$$\check{H}^i(X, \mathcal{F}).$$

For example, consider the case $i = 0$. Given a cover, a cochain is just a collection of sections, (σ_i), $\sigma_i \in \Gamma(U_i, \mathcal{F})$. This cochain is a cocycle if $(\sigma_i - \sigma_j)|_{U_{ij}} = 0$ for every i and j. By the sheaf axiom, this means that there is a global section $\sigma \in \Gamma(X, \mathcal{F})$, so that in fact

$$H^0(\mathcal{U}, \mathcal{F}) = \Gamma(X, \mathcal{F}).$$

It is also sometimes possible to untwist the definition of H^1. A 1-cocycle is precisely the data of a collection

$$(\sigma_{ij}) \in \Gamma(\mathcal{U}, \mathcal{F}),$$

such that

$$\sigma_{ij} - \sigma_{ik} + \sigma_{jk} = 0.$$

In general of course, one does not want to compute these things using limits. The question is how fine does the cover have to be to compute the cohomology? As a first guess one might require that

$$H^i(U_j, \mathcal{F}) = 0,$$

for all j, and $i > 0$. In other words there is no cohomology on each open subset. But this is not enough. One needs instead the slightly stronger condition that

$$H^i(U_I, \mathcal{F}) = 0.$$

Theorem 9.3 (Leray)**.** *If X is a topological space and $\mathcal{F}$ is a sheaf of abelian groups and $\mathcal{U}$ is an open cover such that*

$$H^i(U_I, \mathcal{F}) = 0,$$

for all $i > 0$ and indices I, then in fact the natural map

$$H^i(\mathcal{U}, \mathcal{F}) \simeq \check{H}^i(X, \mathcal{F}),$$

is an isomorphism.

It is in fact not too hard to prove:

Theorem 9.4 (Serre)**.** *Let X be a Noetherian scheme. TFAE*

(1) *X is affine,*
(2) *$H^i(X, \mathcal{F}) = 0$ for all $i > 0$ and all quasi-coherent sheaves,*
(3) *$H^1(X, \mathcal{I}) = 0$ for all coherent ideals $\mathcal{I}$.*

Finally, we need to construct the coboundary maps. Suppose that we are given a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0.$$

We want to define

$$H^i(X, \mathcal{H}) \longrightarrow H^{i+1}(X, \mathcal{F}).$$

Cheating a little, we may assume that we have a commutative diagram with exact rows,

$$\begin{tikzcd}
0 \arrow[r] & C^i(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[d] & C^i(\mathcal{U}, \mathcal{G}) \arrow[r] \arrow[d] & C^i(\mathcal{U}, \mathcal{H}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^{i+1}(\mathcal{U}, \mathcal{F}) \arrow[r] & C^{i+1}(\mathcal{U}, \mathcal{G}) \arrow[r] & C^{i+1}(\mathcal{U}, \mathcal{H}) \arrow[r] & 0.
\end{tikzcd}$$

Suppose we start with an element $t \in H^i(X, \mathcal{H})$. Then t is the image of $t' \in H^i(\mathcal{U}, \mathcal{H})$, for some open cover $\mathcal{U}$. In turn t' is represented by $\tau \in Z^i(\mathcal{U}, \mathcal{H})$. Now we may suppose our cover is sufficiently fine, so that $\tau_I \in \Gamma(U_I, \mathcal{H})$ is the image of $\sigma_I \in \Gamma(U_I, \mathcal{G})$ (and this fixes the cheat). Applying the boundary map, we get $\delta(\sigma) \in C^{i+1}(\mathcal{U}, \mathcal{G})$. Now the image of $\delta(\sigma)$ in $C^{i+1}(\mathcal{U}, \mathcal{H})$ is the same as $\delta(\tau)$, which is zero, as τ is a cocycle. But then by exactness of the bottom rows, we get $\rho \in C^{i+1}(\mathcal{U}, \mathcal{F})$. It is straightforward to check that ρ is a cocycle, so that we get an element $r' \in H^{i+1}(\mathcal{U}, \mathcal{F})$, whence an element r of $H^{i+1}(X, \mathcal{F})$, and that r does not depend on the choice of σ.

Thus sheaf cohomology does exist (at least when X is paracompact, which is not a problem for schemes). Let us calculate the cohomology of projective space.

Theorem 9.5. *Let A be a Noetherian ring. Let $X = \mathbb{P}^r_A$.*

(1) *The natural map $S \longrightarrow \Gamma_*(X, \mathcal{O}_X)$ is an isomorphism.*

(2)
$$H^i(X, \mathcal{O}_X(n)) = 0 \qquad \text{for all} \qquad 0 < i < r \text{ and } n.$$

(3)
$$H^r(X, \mathcal{O}_X(-r-1)) \simeq A.$$

(4) *The natural map*
$$H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-n-r-1)) \longrightarrow H^r(X, \mathcal{O}_X(-r-1)) \simeq A,$$
is a perfect pairing of finitely generated free A-modules.

Proof. Let
$$\mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X(n).$$
Then $\mathcal{F}$ is a quasi-coherent sheaf. Let $\mathcal{U}$ be the standard open affine cover. As every intersection is affine, it follows that we may compute using this cover. Now
$$\Gamma(U_I, \mathcal{F}) = S_{x_I},$$
where
$$x_I = \prod_{i \in I} x_i.$$
Thus Čech cohomology is the cohomology of the complex
$$\prod_{i=0}^{r} S_{x_i} \longrightarrow \prod_{i<j}^{r} S_{x_i x_j} \longrightarrow \dots \longrightarrow S_{x_0 x_1, \dots x_r}.$$
The kernel of the first map is just $H^0(X, \mathcal{F})$, which we already know is S. Now let us turn to $H^r(X, \mathcal{F})$. It is the cokernel of the map
$$\prod_i S_{x_0 x_1 \dots \hat{x}_i \dots x_r} \longrightarrow S_{x_0 x_1 \dots x_r}.$$
The last term is the free A-module with generators all monomials in the Laurent ring (that is, we allow both positive and negative powers).

The image is the set of monomials where x_i has non-negative exponent. Thus the cokernel is naturally identified with the free A-module generated by arbitrary products of reciprocals x_i^{-1},

$$\{ x_0^{l_0} x_1^{l_1} \dots x_r^{l_r} \mid l_i < 0 \}.$$

The grading is then given by

$$l = \sum_{i=0}^{r} l_i.$$

In particular

$$H^r(X, \mathcal{O}_X(-r-1)),$$

is the free A-module with generator $x_0^{-1} x_1^{-1} \dots x_r^{-1}$. Hence (3).

To define a pairing, we declare

$$x_0^{l_0} x_1^{l_1} \dots x_r^{l_r},$$

to be the dual of

$$x_0^{m_0} x_1^{m_1} \dots x_r^{m_r} = x_0^{-1-l_0} x_1^{-1-l_1} \dots x_r^{-1-l_r}.$$

As $m_i \geq 0$ if and only if $l_i < 0$ it is straightforward to check that this gives a perfect pairing. Hence (4).

It remains to prove (2). If we localise the complex above with respect to x_r, we get a complex which computes $\mathcal{F}|_{U_r}$, which is zero in positive degree, as U_r is affine. Thus

$$H^i(X, \mathcal{F})_{x_r} = 0,$$

for $i > 0$ so that every element of $H^i(X, \mathcal{F})$ is annihilated by some power of x_r.

To finish the proof, we will show that multiplication by x_r induces an inclusion of cohomology. We proceed by induction on the dimension. Suppose that $r > 1$ and let $Y \simeq \mathbb{P}^{r-1}_A$ be the hyperplane $x_r = 0$. Then

$$\mathcal{I}_Y = \mathcal{O}_X(-Y) = \mathcal{O}_X(-1).$$

Thus there are short exact sequences

$$0 \longrightarrow \mathcal{O}_X(n-1) \longrightarrow \mathcal{O}_X(n) \longrightarrow \mathcal{O}_Y(n) \longrightarrow 0.$$

Now $H^i(Y, \mathcal{O}_Y(n)) = 0$ for $0 < i < r-1$ and the natural restriction map

$$H^0(X, \mathcal{O}_X(n)) \longrightarrow H^0(Y, \mathcal{O}_Y(n)),$$

is surjective (every polynomial of degree n on Y is the restriction of a polynomial of degree n on X). Thus

$$H^i(X, \mathcal{O}_X(n-1)) \simeq H^i(X, \mathcal{O}_X(n)),$$

for $0 < i < r-1$, and even if $i = r-1$, then we get an injective map. But this map is the one induced by multiplication by x_r. $\square$

Theorem 9.6 (Serre vanishing)**.** *Let X be a projective variety over a Noetherian ring and let $\mathcal{O}_X(1)$ be a very ample line bundle on X. Let $\mathcal{F}$ be a coherent sheaf.*

(1) *$H^i(X, \mathcal{F})$ are finitely generated A-modules.*
(2) *There is an integer n_0 such that $H^i(X, \mathcal{F}(n)) = 0$ for all $n \geq n_0$ and $i > 0$.*

Proof. By assumption there is an immersion $i\colon X \longrightarrow \mathbb{P}^r_A$ such that $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}^r_A}(1)$. As X is projective, it is proper and so i is a closed immersion. If $\mathcal{G} = i_*\mathcal{F}$ then

$$H^i(\mathbb{P}^r_A, \mathcal{G}) \simeq H^i(X, \mathcal{F}).$$

Replacing X by $\mathbb{P}^r_A$ and $\mathcal{F}$ by $\mathcal{G}$ we may assume that $X = \mathbb{P}^r_A$.

If $\mathcal{F} = \mathcal{O}_X(q)$ then the result is given by (9.5). Thus the result also holds is $\mathcal{F}$ is a direct sum of invertible sheaves. The general case proceeds by descending induction on i. Now

$$H^i(X, \mathcal{F}) = 0,$$

if $i > r$ (clear, if we use Čech cohomology). On the other hand, $\mathcal{F}$ is a quotient of a direct sum $\mathcal{E}$ of invertible sheaves. Thus there is an exact sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\mathcal{R}$ is coherent. Twisting by $\mathcal{O}_X(n)$ we get

$$0 \longrightarrow \mathcal{R}(n) \longrightarrow \mathcal{E}(n) \longrightarrow \mathcal{F}(n) \longrightarrow 0.$$

Taking the long exact sequence of cohomology, we get isomorphisms

$$H^i(X, \mathcal{F}(n)) \simeq H^{i+1}(X, \mathcal{R}(n)),$$

for n large enough, and we are done by descending induction on i. $\square$

Theorem 9.7. *Let A be a Noetherian ring and let X be a proper scheme over A. Let $\mathcal{L}$ be an invertible sheaf on X. TFAE*

(1) *$\mathcal{L}$ is ample.*
(2) *For every coherent sheaf $\mathcal{F}$ on X there is an integer n_0 such that*

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0,$$

for $n > n_0$.

Proof. (1) implies (2) is proved using the division algorithm, as in the proof of (7.7).

Now suppose that (2) holds. Let $\mathcal{F}$ be a coherent sheaf. Let $p \in X$ be a closed point. Consider the short exact sequence

$$0 \longrightarrow \mathcal{I}_p\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F} \otimes \mathcal{O}_p \longrightarrow 0,$$

where $\mathcal{I}_p$ is the ideal sheaf of p. If we tensor this exact sequence with $\mathcal{L}^n$ we get an exact sequence

$$0 \longrightarrow \mathcal{I}_p\mathcal{F} \otimes \mathcal{L}^n \longrightarrow \mathcal{F} \otimes \mathcal{L}^n \longrightarrow \mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{O}_p \longrightarrow 0.$$

By hypotheses we can find n_0 such that

$$H^1(X, \mathcal{I}_p\mathcal{F} \otimes \mathcal{L}^n) = 0,$$

for all $n \geq n_0$. It follows that the natural map

$$H^0(X, \mathcal{F} \otimes \mathcal{L}^n) \longrightarrow H^0(X, \mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{O}_p),$$

is surjective, for all $n \geq n_0$. It follows by Nakayama's lemma applied to the local ring $\mathcal{O}_{X,p}$ that that the stalk of $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections. As $\mathcal{F}$ is a coherent sheaf, for each integer $n \neq n_0$ there is an open subset U, depending on n, such that sections of $H^0(X, \mathcal{F} \otimes \mathcal{L})$ generate the sheaf at every point of U.

If we take $\mathcal{L} = \mathcal{O}_X$ it follows that there is an integer n_1 such that $\mathcal{L}^{n_1}$ is generated by global sections over an open neighbourhood V of p. For each $0 \leq r \leq n_1 - 1$ we may find U_r such that $\mathcal{F} \otimes \mathcal{L}^{n_0+r}$ is generated by global sections over U_r. Now let

$$U_p = V \cap U_0 \cap U_1 \cap \cdots \cap U_{n_1-1}.$$

Then

$$\mathcal{F} \otimes \mathcal{L}^n = (\mathcal{F} \otimes \mathcal{L}^{n_0+r}) \otimes (\mathcal{L}^{n_1})^m,$$

is generated by global sections over the whole of U_p for all $n \neq n_0$.

Now use compactness of X to conclude that we can cover X by finitely many U_p. $\square$

Theorem 9.8 (Serre duality)**.** *Let X be a smooth projective variety of dimension n over an algebraically closed field. Then there is an invertible sheaf ω_X such that*

(1) $h^n(X, \omega_X) = 1$.
(2) *Given any other invertible sheaf $\mathcal{L}$ there is a perfect pairing*

$$H^i(X, \mathcal{L}) \times H^{n-i}(X, \omega_X \otimes \mathcal{L}^*) \longrightarrow H^n(X, \omega_X).$$

Example 9.9. *Let $X = \mathbb{P}^r_k$. Then $\omega_X = \mathcal{O}_X(-r-1)$ is a dualising sheaf.*

In fact, on any smooth projective variety, the dualising sheaf is constructed as the determinant of the cotangent bundle, which is a locally free sheaf. To construct the cotangent bundle, let $i\colon X \longrightarrow X \times X$ be the diagonal embedding. Let $\mathcal{I}$ be the ideal sheaf of the diagonal and let

$$\Omega^1_X = i^* \frac{\mathcal{I}}{\mathcal{I}^2}.$$

Ω^1_X is the dual of the tangent bundle. Ω^1_X is a locally free sheaf of rank n, known as the sheaf of Kähler differentials. The determinant sheaf is then the dualising sheaf,

$$\omega_X = \wedge^n \Omega^1_X.$$

This expresses a remarkable coincidence between the dualising sheaf, which is something defined in terms of sheaf cohomology and the determinant of the sheaf of Kähler differentials, which is something which comes from calculus on the variety.

Theorem 9.10. *Let $X = X(F)$ be a toric variety over $\mathbb{C}$ and let D be a T-Cartier divisor. Given $u \in M$ let*

$$Z(u) = \{\, v \in |F| \,|\, \langle u, v\rangle \geq \psi_D(v) \,\}.$$

Then

$$H^p(X, \mathcal{O}_X(D)) = \bigoplus_{u\in M} H^p(X, \mathcal{O}_X(D))_u \qquad \text{where} \qquad H^p(X, \mathcal{O}_X(D))_u = H^p_{Z(u)}(|F|).$$

Some explanation is in order. Note that the cohomology groups of X are naturally graded by M. (9.10) identifies the graded pieces.

$$H^p_{Z(u)}(|F|) = H^p(|F|, |F| - Z(u), \mathbb{C}).$$

denotes local cohomology. This comes with a long exact sequence for the pair. If X is an affine toric variety then both $|F|$ and $Z(u)$ are convex and the local cohomology vanishes. More generally, if D is ample, then then both $|F|$ and $Z(u)$ are convex and the local cohomology vanishes. This gives a slightly stronger result than Serre vanishing in the case of an arbitrary variety.

It is interesting to calculate the dualising sheaf in the case of a smooth toric variety. First of all note that the dualising sheaf is a line bundle, so that $\omega_X = \mathcal{O}_X(K_X)$, for some divisor K_X, which is called the **canonical divisor**. Note that the canonical divisor is only defined up to linear equivalence.

To calculate the canonical divisor, we need to write down a rational (or meromorphic in the case of $\mathbb{C}$) differential form. Note that if

$z_1, z_2, \dots, z_n$ are coordinates on the torus then

$$\frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge \cdots \wedge \frac{dz_n}{z_n},$$

is invariant under the action of the torus, so that the associated divisor is supported on the invariant divisor. With a little bit of work one can show that this rational form has a simple pole along every invaraint divisor, that is

$$K_X + D \sim 0,$$

where D is a sum of the invariant divisors. For example,

$$-K_{\mathbb{P}^n} = H_0 + H_1 + \cdots + H_n \sim (n+1)H,$$

as expected.

Even if X is not smooth, it is possible to define the canonical divisor. Suppose that X is normal, so that the singular locus has codimension at least two. Let U be the smooth locus and let K_U be the canonical divisor of U. Let K_X be the divisor obtained by taking the closure of the components of K_U. Note that K_X is only defined as a Weil divisor in this case.