7. Linear systems

First a word about the base scheme. We would like to work in enough generality to cover the general case. On the other hand, it takes some work to state properly the general results if one works over an arbitrary scheme \( S \). As a compromise we work over an arbitrary affine variety \( S = \text{Spec } A \). As most statements are local on the base, we don’t lose any generality. It is customary to say \( X \) is a scheme over a ring \( A \), as shorthand for \( X \) is a scheme over the corresponding scheme \( S = \text{Spec } A \).

**Theorem 7.1.** Let \( X \) be a scheme over a ring \( A \).

1. If \( \phi : X \to \mathbb{P}^n_A \) is an \( A \)-morphism then \( L = \phi^* \mathcal{O}_{\mathbb{P}^n_A}(1) \) is an invertible sheaf on \( X \), which is generated by the global sections \( s_0, s_1, \ldots, s_n \), where \( s_i = \phi^* x_i \).

2. If \( L \) is an invertible sheaf on \( X \), which is generated by the global sections \( s_0, s_1, \ldots, s_n \), then there is a unique \( A \)-morphism \( \phi : X \to \mathbb{P}^n_A \) such that \( L = \phi^* \mathcal{O}_{\mathbb{P}^n_A}(1) \) and \( s_i = \phi^* x_i \).

**Proof.** It is clear that \( L \) is an invertible sheaf. Since \( x_0, x_1, \ldots, x_n \) generate the ring \( A[x_0, x_1, \ldots, x_n] \), it follows that \( x_0, x_1, \ldots, x_n \) generate the sheaf \( \mathcal{O}_{\mathbb{P}^n_A}(1) \). Thus \( s_0, s_1, \ldots, s_n \) generate \( L \). Hence (1).

Now suppose that \( L \) is an invertible sheaf generated by \( s_0, s_1, \ldots, s_n \). Let

\[ X_i = \{ p \in X | s_i \notin \mathfrak{m}_p L_p \}. \]

Then \( X_i \) is an open subset of \( X \) and the sets \( X_0, X_1, \ldots, X_n \) cover \( X \). Define a morphism

\[ \phi_i : X_i \to U_i, \]

where \( U_i \) is the standard open subset of \( \mathbb{P}^n_A \), as follows: Since

\[ U_i = \text{Spec } A[y_0, y_1, \ldots, y_n], \]

where \( y_j = x_j / x_i \), is affine, it suffices to give a ring homomorphism

\[ A[y_0, y_1, \ldots, y_n] \to \Gamma(X_i, \mathcal{O}_{X_i}). \]

We send \( y_j \) to \( s_j / s_i \), and extend by linearity. The key observation is that the ratio is a well-defined element of \( \mathcal{O}_{X_i} \), which does not depend on the choice of isomorphism \( L|_{X_i} \simeq \mathcal{O}_{X_i} \).

It is then straightforward to check that the set of morphisms \( \{ \phi_i \} \) glues to a morphism \( \phi \) with the given properties. \( \square \)

**Example 7.2.** Let \( X = \mathbb{P}^1_k, A = k, L = \mathcal{O}_{\mathbb{P}^1_k}(2) \).

In this case, global sections of \( L \) are generated by \( S^2, ST \) and \( T^2 \). This morphism is represented globally by

\[ [S : T] \to [S^2 : ST : T^2]. \]
The image is the conic \( XZ = Y^2 \) inside \( \mathbb{P}^2_k \).

More generally one can map \( \mathbb{P}^1_k \) into \( \mathbb{P}^n_k \) by the invertible sheaf \( \mathcal{O}_{\mathbb{P}^1_k}(n) \).

More generally still, one can map \( \mathbb{P}^m_k \) into \( \mathbb{P}^n_k \) using the invertible sheaf \( \mathcal{O}_{\mathbb{P}^m_k}(1) \).

**Corollary 7.3.**

\[
\text{Aut}(\mathbb{P}^n_k) \simeq \text{PGL}(n + 1, k).
\]

**Proof.** First note that \( \text{PGL}(n + 1, k) \) acts naturally on \( \mathbb{P}^n_k \) and that this action is faithful.

Now suppose that \( \phi \in \text{Aut}(\mathbb{P}^n_k) \). Let \( \mathcal{L} = \phi^* \mathcal{O}_{\mathbb{P}^n_k}(1) \). Since \( \text{Pic}(\mathbb{P}^n_k) \simeq \mathbb{Z} \) is generated by \( \mathcal{O}_{\mathbb{P}^n_k}(1) \), it follows that \( \mathcal{L} \simeq \mathcal{O}_{\mathbb{P}^n_k}(\pm 1) \). As \( \mathcal{L} \) is globally generated, we must have \( \mathcal{L} \simeq \mathcal{O}_{\mathbb{P}^n_k}(1) \). Let \( s_i = \phi^* x_i \). Then \( s_0, s_1, \ldots, s_n \) is a basis for the \( k \)-vector space \( H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1)) \). But then there is a matrix

\[
A = (a_{ij}) \in \text{GL}(n + 1, k) \quad \text{such that} \quad s_i = \sum_{ij} a_{ij} x_j.
\]

Since the morphism \( \phi \) is determined by \( s_0, s_1, \ldots, s_n \), it follows that \( \phi \) is determined by the class of \( A \) in \( \text{GL}(n + 1, k) \). \( \square \)

**Lemma 7.4.** Let \( \phi: X \longrightarrow \mathbb{P}^n_A \) be an \( A \)-morphism. Then \( \phi \) is a closed immersion if and only if

1. \( X_i = X_{s_i} \) is affine, and
2. the natural map of rings

\[
A[y_0, y_1, \ldots, y_n] \longrightarrow \Gamma(X_i, \mathcal{O}_{X_i}) \quad \text{which sends} \quad y_i \longrightarrow \frac{s_i}{s_j},
\]

is surjective.

**Proof.** Suppose that \( \phi \) is a closed immersion. Then \( X_i \) is isomorphic to \( \phi(X) \cap U_i \), a closed subscheme of affine space. Thus \( X_i \) is affine. Hence (1) and (2) follows as we have surjectivity on all of the localisations.

Now suppose that (1) and (2) hold. Then \( X_i \) is a closed subscheme of \( U_i \) and so \( X \) is a closed subscheme of \( \mathbb{P}^n_A \). \( \square \)

**Theorem 7.5.** Let \( X \) be a projective scheme over an algebraically closed field \( k \) and let \( \phi: X \longrightarrow \mathbb{P}^n_k \) be a morphism over \( k \), which is given by an invertible sheaf \( \mathcal{L} \) and global sections \( s_0, s_1, \ldots, s_n \) which generate \( \mathcal{L} \). Let \( V \subset \Gamma(X, \mathcal{L}) \) be the space spanned by the sections.

Then \( \phi \) is a closed immersion if and only if

1. \( V \) separates points: that is, given \( p \) and \( q \) in \( X \) there is \( \sigma \in V \) such that \( \sigma \in \mathfrak{m}_p \mathcal{L}_p \) but \( \sigma \notin \mathfrak{m}_q \mathcal{L}_q \).
(2) \textbf{V separates tangent vectors}: that is, given \( p \in X \) the set
\[
\{ \sigma \in V \mid \sigma \in m_p L_p \},
\]
spans \( m_p L_p / m_p^2 L_p \).

\textbf{Proof.} Suppose that \( \phi \) is a closed immersion. Then we might as well consider \( X \subset \mathbb{P}^n_k \) as a closed subscheme. In this case (1) is clear. Just pick a linear function on the whole of \( \mathbb{P}^n_k \) which vanishes at \( p \) but not at \( q \) (equivalently pick a hyperplane which contains \( p \) but not \( q \)). Similarly linear functions on \( \mathbb{P}^n_k \) separate tangent vectors on the whole of projective space, so they certainly separate on \( X \).

Now suppose that (1) and (2) hold. Then \( \phi \) is clearly injective. Since \( X \) is proper over \( \text{Spec} \, k \) and \( \mathbb{P}^n_k \) is separated over \( \text{Spec} \, k \) it follows that \( \phi \) is proper. In particular, \( \phi(X) \) is closed and \( \phi \) is a homeomorphism onto \( \phi(X) \). It remains to show that the map on stalks
\[
O_{\mathbb{P}^n_k, p} \longrightarrow O_{X, x},
\]
is surjective. But the same piece of commutative algebra as we used in the proof of the inverse function theorem, works here. \( \square \)

\textbf{Definition 7.6.} \textit{Let} \( X \) \textit{be a noetherian scheme. We say that an invertible sheaf} \( L \) \textit{is ample if for every coherent sheaf} \( F \) \textit{there is an integer} \( n_0 > 0 \) \textit{such that} \( F \otimes L^n \) \textit{is globally generated, for all} \( n \geq n_0 \).

\textbf{Lemma 7.7.} \textit{Let} \( L \) \textit{be an invertible sheaf on a Noetherian scheme. The following are equivalent:}

1. \( L \) \textit{is ample.}
2. \( L^m \) \textit{is ample for all} \( m > 0 \).
3. \( L^m \) \textit{is ample for some} \( m > 0 \).

\textbf{Proof.} (1) implies (2) implies (3) is clear.

So assume that \( M = L^m \) is ample and let \( F \) be a coherent sheaf. For each \( 0 \leq i \leq m - 1 \), let \( F_i = F \otimes L^i \). By assumption there is an integer \( n_i \) such that \( F_i \otimes M^n \) is globally generated for all \( n \geq n_i \). Let \( n_0 \) be the maximum of the \( n_i \). If \( n \geq n_0 m \), then we may write \( n = qm + i \), where \( 0 \leq i \leq m - 1 \) and \( q \geq n_0 \geq n_i \).

But then
\[
F \otimes L^m = F_i \otimes M^q,
\]
which is globally generated. \( \square \)

\textbf{Theorem 7.8.} \textit{Let} \( X \) \textit{be a scheme of finite type over a Noetherian ring} \( A \) \textit{and let} \( L \) \textit{be an invertible sheaf on} \( X \).

\textit{Then} \( L \) \textit{is ample if and only if} \( L^m \) \textit{is very ample for some} \( m > 0 \).
Proof. Suppose that \( \mathcal{L}^m \) is very ample. Then there is an immersion \( X \subset \mathbb{P}^r_A \), for some positive integer \( r \), and \( \mathcal{L}^m = \mathcal{O}_X(1) \). Let \( X \) be the closure. If \( \mathcal{F} \) is any coherent sheaf on \( X \) then there is a coherent sheaf \( \mathcal{F}^n \) on \( X \), such that \( \mathcal{F} = \mathcal{F}^n|_X \). By Serre’s result, \( \mathcal{F}(k) \) is globally generated for some positive integer \( k \). It follows that \( \mathcal{F}(k) \) is globally generated, so that \( \mathcal{L}^m \) is ample, and the result follows by (7.7).

Conversely, suppose that \( \mathcal{L} \) is ample. Given \( p \in X \), pick an open affine neighbourhood \( U \) of \( p \) so that \( \mathcal{L}|_U \) is free. Let \( Y = X - U \), give it the reduced induced structure, with ideal sheaf \( \mathcal{I} \). Then \( \mathcal{I} \) is coherent.

By compactness, we may cover \( X \) by finitely many such open affines and we may assume that \( n \) is fixed. Replacing \( \mathcal{L} \) by \( \mathcal{L}^n \) we may assume that \( n = 1 \).

Then there are global sections \( s_1, s_2, \ldots, s_k \in H^0(X, \mathcal{L}) \) such that \( U_i = U_{s_i} \) is an open affine cover.

Since \( X \) is of finite type, each \( \mathcal{B}_i = H^0(U_i, \mathcal{O}_{U_i}) \) is a finitely generated \( A \)-algebra. Pick generators \( b_{ij} \). Then \( s^n b_{ij} \) lifts to \( s_{ij} \in H^0(X, \mathcal{L}^n) \), for some positive integer \( n \). Again we might as well assume \( n = 1 \).

Now let \( \mathbb{P}^N_A \) be the projective space with coordinates \( x_1, x_2, \ldots, x_k \) and \( x_{ij} \). Locally we can define a map on each \( U_i \) to the standard open affine, by the obvious rule, and it is standard to check that this glues to an immersion.

\[ \square \]

Definition 7.9. Let \( \mathcal{L} \) be an invertible sheaf on a smooth projective variety over an algebraically closed field. Let \( s \in H^0(X, \mathcal{L}) \). The divisor \( \langle s \rangle \) of zeroes of \( s \) is defined as follows: by assumption we may cover \( X \) by open subsets \( U_i \) over which we may identify \( s|_{U_i} \) with \( f_i \in \mathcal{O}_{U_i} \). The defines a Cartier divisor \( \{(U_i, f_i)\} \).

It is a simple matter to check that the Cartier divisor does not depend on our choice of trivialisations. Note that as \( X \) is smooth the Cartier divisor may safely be identified with the corresponding Weil divisor.

Lemma 7.10. Let \( X \) be a smooth projective variety over an algebraically closed field. Let \( D_0 \) be a divisor and let \( \mathcal{L} = \mathcal{O}_X(D_0) \).

(1) If \( s \in H^0(X, \mathcal{L}) \), \( s \neq 0 \) then \( \langle s \rangle \sim D_0 \).

(2) If \( D \geq 0 \) and \( D \sim D_0 \) then there is a global section \( s \in H^0(X, \mathcal{L}) \) such that \( D = \langle s \rangle \).

(3) If \( s_i \in H^0(X, \mathcal{L}) \), \( i = 1 \) and 2, are two global sections then \( \langle s_1 \rangle = \langle s_2 \rangle \) if and only if \( s_2 = \lambda s_1 \) where \( \lambda \in k^* \).

Proof. As \( \mathcal{O}_X(D_0) \subset \mathcal{K} \), the section \( s \) corresponds to a rational function \( f \). If \( D_0 \) is the Cartier divisor \( \{(U_i, f_i)\} \) then \( \mathcal{O}_X(D_0) \) is locally
generated by $f_i^{-1}$ so that multiplication by $f_i$ induces an isomorphism with $\mathcal{O}_{U_i}$. $D$ is then locally defined by $ff_i$. But then

$$D = D_0 + (f).$$

Hence (1).

Now suppose that $D > 0$ and $D = D_0 + (f)$. Then $(f) \geq -D_0$. Hence

$$f \in H^0(X, \mathcal{O}_X(D_0)) \subset H^0(X, \mathcal{K}) = K(X),$$

and the divisor of zeroes of $f$ is $D$. This is (2).

Now suppose that $(s_1) = (s_2)$. Then

$$D_0 + (f_1) = (s_1) = (s_2) = D_0 + (f_2).$$

Cancelling, we get that $(f_1) = (f_2)$ and the rational function $f_1/f_2$ has no zeroes nor poles. Since $X$ is a projective variety, $f_1/f_2 = \lambda$, a constant. □

**Definition 7.11.** Let $D_0$ be a divisor. The complete linear system associated to $D_0$ is the set

$$|D_0| = \{ D \in \text{Div}(X) \mid D \geq 0, D \sim D_0 \}.$$  

We have seen that

$$|D| = \mathbb{P}(H^0(X, \mathcal{O}_X(D_0))).$$

Thus $|D|$ is naturally a projective space.

**Definition 7.12.** A linear system is any linear subspace of a complete linear system $|D_0|$.

In other words, a linear system corresponds to a linear subspace, $V \subset H^0(X, \mathcal{O}_X(D_0))$. We will then write

$$|V| = \{ D \in |D_0| \mid D = (s), s \in V \} \simeq \mathbb{P}(V) \subset \mathbb{P}(H^0(X, \mathcal{O}_X(D_0))).$$

**Definition 7.13.** Let $|V|$ be a linear system. The base locus of $|V|$ is the intersection of the elements of $|V|$.

**Lemma 7.14.** Let $X$ be a smooth projective variety over an algebraically closed field, and let $|V| \subset |D_0|$ be a linear system.

$V$ generates $\mathcal{O}_X(D_0)$ if and only if $|V|$ is base point free.

**Proof.** If $V$ generates $\mathcal{O}_X(D_0)$ then for every point $x \in X$ we may find an element $\sigma \in V$ such that $\sigma(x) \neq 0$. But then $D = (\sigma)$ does not contain $x$, and so the base locus is empty.

Conversely suppose that the base locus is empty. The locus where $V$ is not generated $\mathcal{O}_X(D_0)$ is a closed subset $Z$ of $X$. Pick $x \in Z$ a closed point. By assumption we may find $D \in |V|$ such that $x \notin D$. But then
if $D = (\sigma)$, $\sigma(x) \neq 0$ and $\sigma$ generates the stalk $L_x$, a contradiction. Thus $Z$ is empty and $O_X(D_0)$ is globally generated. \qed

**Example 7.15.** Consider $O_{\mathbb{P}^2}(4)$. The complete linear system $|4p|$ defines a morphism into $\mathbb{P}^4$, where $p = [0 : 1]$ and $q = [1 : 0]$, given by $\mathbb{P}^1 \rightarrow \mathbb{P}^4$, $[S : T] \rightarrow [S^4 : ST^3 : S^3T^2 : ST^3 : T^4]$. If we project from $[0 : 0 : 1 : 0 : 0]$ we will get a morphism into $\mathbb{P}^3$, $[S : T] \rightarrow [S^4 : ST^3 : T^4]$. This corresponds to the sublinear system spanned by $4p, 3p + q, p + 3q, 4q$.

Consider $O_{\mathbb{P}^2}(2)$ and the corresponding complete linear system. The map associated to this linear system is the Veronese embedding $\mathbb{P}^2 \rightarrow \mathbb{P}^5$, $[X : Y : Z] \rightarrow [X^2 : Y^2 : Z^2 : YZ : XZ : XY]$.

Note also the notion of separating points and tangent directions becomes a little clearer in this more geometric setting. Separating points means that given $x$ and $y \in X$, we can find $D \in |V|$ such that $x \in D$ and $y \notin D$. Separating tangent vectors means that given any irreducible length two zero dimensional scheme $z$, with support $x$, we can find $D \in |V|$ such that $x \in D$ but $z$ is not contained in $D$. In fact the condition about separating tangent vectors is really the limiting case of separating points.

Thinking in terms of linear systems also presents an inductive approach to proving global generation. Suppose that we consider the complete linear system $|D|$. Suppose that we can find $Y \in |D|$. Then the base locus of $|D|$ is supported on $Y$. On the other hand suppose that $\mathcal{I}$ is the ideal sheaf of $Y$ in $X$. Then there is an exact sequence

$$0 \rightarrow \mathcal{I} \rightarrow O_X \rightarrow O_Y \rightarrow 0.$$ 

As $X$ is smooth $D$ is Cartier and $O_X(D)$ is an invertible sheaf. Tensoring by locally free preserves exactness, so there are short exact sequences,

$$0 \rightarrow \mathcal{I}(mD) \rightarrow O_X(mD) \rightarrow O_Y(mD) \rightarrow 0.$$ 

Taking global sections, we get

$$0 \rightarrow H^0(X, \mathcal{I}(mD)) \rightarrow H^0(X, O_X(mD)) \rightarrow H^0(Y, O_Y(mD)).$$ 

At the level of linear systems there is therefore a linear map

$$|D| \rightarrow |D|_Y.$$ 

It is interesting to see what happens for toric varieties. Suppose that $X = X(F)$ is the toric variety associated to the fan $F \subset N_\mathbb{R}$. Recall that we can associate to a $T$-Cartier divisor $D = \sum a_iD_i$, a continuous piecewise linear function

$$\phi_D : |F| \rightarrow \mathbb{R},$$
where $|F| \subset N_R$ is the support of the fan, which is specified by the rule that $\phi_D(v_i) = -a_i$.

We can also associate to $D$ a rational polyhedron

$$P_D = \{ u \in M_R \mid \langle u, v_i \rangle \geq -a_i \ \forall \ v_i \} = \{ u \in M_R \mid u \geq \phi_D \}.$$

Lemma 7.16. If $X$ is a toric variety and $D$ is $T$-Cartier then

$$H^0(X, \mathcal{O}_X(D)) = \bigoplus_{u \in P_D \cap M} k \cdot \chi^u.$$  

Proof. Suppose that $\sigma \in F$ is a cone. Then, we have already seen that

$$H^0(U_\sigma, \mathcal{O}_{U_\sigma}(D)) = \bigoplus_{u \in P_D(\sigma) \cap M} k \cdot \chi^u,$$

where

$$P_D(\sigma) = \{ u \in M_R \mid \langle u, v_i \rangle \geq -a_i \ \forall \ v_i \in \sigma \}.$$

These identifications are compatible on overlaps. Since

$$H^0(X, \mathcal{O}_X(D)) = \bigcap_{\sigma \in F} H^0(U_\sigma, \mathcal{O}_{U_\sigma}(D))$$

and

$$P_D = \bigcap_{\sigma \in F} P_D(\sigma),$$

the result is clear. \qed

It is interesting to compute some examples. First, suppose we consider $\mathbb{P}^1$. A $T$-Cartier divisor is a sum $ap+bq$ ($p$ and $q$ the fixed points). The corresponding function is

$$\phi(x) = \begin{cases} -ax & x > 0 \\ -bx & x < 0. \end{cases}$$

The corresponding polytope is the interval

$$[-a, b] \subset \mathbb{R} = M_\mathbb{R}.$$

There are $a+b+1$ integral points, corresponding to the fact that there are $a+b+1$ monomials of degree $a+b$. For $\mathbb{P}^2$ and $dD_3$, $P_D$ is the convex hull of $(0,0), (-d,0)$ and $(0,-d)$. The number of integral points is

$$\frac{(d+1)^2}{2} + \frac{d+1}{2} = \frac{(d+2)(d+1)}{2},$$

which is the usual formula.
Let \( D \) be a Cartier divisor on a toric variety \( X = X(F) \) given by a fan \( F \). It is interesting to consider when the complete linear system \( |D| \) is base point free. Since any Cartier divisor is linearly equivalent to a \( T \)-Cartier divisor, we might as well suppose that \( D = \sum a_i D_i \) is \( T \)-Cartier. Note that the base locus of the complete linear system of any Cartier divisor is invariant under the action of the torus. Since there are only finitely many orbits, it suffices to show that for each cone \( \sigma \in F \) the point \( x_\sigma \in U_\sigma \) is not in the base locus. It is also clear that if \( x_\sigma \) is not in the base locus of \( |D| \) then in fact one can find a \( T \)-Cartier divisor \( D' \in |D| \) which does not contain \( x_\sigma \). Equivalently we can find \( u \in M \) such that
\[
\langle u, v_i \rangle \geq -a_i,
\]
with strict equality if \( v_i \in \sigma \). The interesting thing is that we can reinterpret this condition using \( \phi_D \).

**Definition 7.17.** The function \( \phi: V \longrightarrow \mathbb{R} \) is upper convex if

\[
\phi(\lambda v + (1 - \lambda)w) \geq \lambda \phi(v) + (1 - \lambda)w \quad \forall v, w \in V.
\]

When we have a fan \( F \) and \( \phi \) is linear on each cone \( \sigma \), then \( \phi \) is called strictly upper convex if the linear functions \( u(\sigma) \) and \( u(\sigma') \) are different, for different maximal cones \( \sigma \) and \( \sigma' \).

**Theorem 7.18.** Let \( X = X(F) \) be the toric variety associated to a \( T \)-Cartier divisor \( D \).

Then

1. \( |D| \) is base point free if and only if \( \psi_D \) is upper convex.
2. \( D \) is very ample if and only if \( \psi_D \) is strictly upper convex and the semigroup \( S_\sigma \) is generated by
   \[
   \{ u - u(\sigma) \mid u \in P_D \cap M \}.
   \]

**Proof.** (1) follows from the remarks above. (2) is proved in Fulton’s book. \( \square \)

For example if \( X = \mathbb{P}^1 \) and
\[
\phi(x) = \begin{cases} 
-ax & x > 0 \\
-bx & x < 0.
\end{cases}
\]
so that \( D = ap + bq \) then \( \phi \) is upper convex if and only if \( a + b \geq 0 \) in which case \( D \) is base point free. \( D \) is very ample if and only if \( a + b > 0 \). When \( \phi \) is continuous and linear on each cone \( \sigma \), we may restate the upper convex condition as saying that the graph of \( \phi \) lies under the graph of \( u(\sigma) \). It is strictly upper convex if it lies strictly under the graph of \( u(\sigma) \), for all \( n \)-dimensional cones \( \sigma \).
Using this, it is altogether too easy to give an example of a smooth proper toric variety which is not projective. Let $F \subset \mathbb{N}_R = \mathbb{R}^3$ given by the edges $v_1 = -e_1$, $v_2 = -e_2$, $v_3 = -e_3$, $v_4 = e_1 + e_2 + e_3$, $v_5 = v_3 + v_4$, $v_6 = v_1 + v_4$ and $v_7 = v_2 + v_4$. Now connect $v_1$ to $v_5$, $v_3$ to $v_7$ and $v_2$ to $v_6$ and $v_5$ to $v_6$, $v_6$ to $v_7$ and $v_7$ to $v_5$.

It is not hard to check that $X$ is smooth and proper (proper translates to the statement that the support $|F|$ of the fan is the whole of $\mathbb{N}_R$). Suppose that $\psi$ is strictly upper convex. Let $w$ be the midpoint of the line connecting $v_1$ and $v_5$. Then

$$w = \frac{v_1 + v_5}{2} = \frac{v_3 + v_6}{2}.$$

Since $v_1$ and $v_5$ belong to the same maximal cone, $\psi$ is linear on the line connecting them. In particular

$$\psi(w) = \psi\left(\frac{v_1 + v_5}{2}\right) = \frac{1}{2} \psi(v_1) + \frac{1}{2} \psi(v_5).$$

Since $v_1$, $v_5$ and $v_3$ belong to the same cone and $v_6$ does not, by strict convexity,

$$\psi(w) = \psi\left(\frac{v_3 + v_6}{2}\right) > \frac{1}{2} \psi(v_3) + \frac{1}{2} \psi(v_6).$$

Putting all of this together, we get

$$\psi(v_1) + \psi(v_5) > \psi(v_2) + \psi(v_6).$$

By symmetry

$$\psi(v_1) + \psi(v_5) > \psi(v_3) + \psi(v_6)$$
$$\psi(v_2) + \psi(v_6) > \psi(v_1) + \psi(v_7)$$
$$\psi(v_3) + \psi(v_7) > \psi(v_2) + \psi(v_5).$$

But adding up these three inequalities gives a contradiction.