2. Toric varieties

First some stuff about algebraic groups:

Definition 2.1. *Let G be a group. We say that G is an* ***algebraic group*** *if G is a quasi-projective variety and the two maps $m\colon G\times G \longrightarrow G$ and $i\colon G \longrightarrow G$, where m is multiplication and i is the inverse map, are both morphisms.*

It is easy to give examples of algebraic groups. Consider the group $G = \mathrm{GL}_n(K)$. In this case G is an open subset of $\mathbb{A}^{n^2}$, the complement of the zero locus of the determinant, which expands to a polynomial. Matrix multiplication is obviously a morphism, and the inverse map is a morphism by Cramer's rule. Note that there are then many obvious algebraic subgroups; the orthogonal groups, special linear group and so on. Clearly $\mathrm{PGL}_n(K)$ is also an algebraic group; indeed the quotient of an algebraic group by a closed normal subgroup is an algebraic group. All of these are affine algebraic groups.

Definition 2.2. *Let G be an algebraic group. If G is affine then we say that G is a* ***linear algebraic group****. If G is projective and connected then we say that G is an* ***abelian variety****.*

Note that any finite group is an algebraic group (both affine and projective). It turns out that any affine group is always a subgroup of a matrix group, so that the notation makes sense.

Definition 2.3. *The group $\mathbb{G}_m$ is $\mathrm{GL}_1(K)$. The group $\mathbb{G}_a$ is the subgroup of $\mathrm{GL}_2(K)$ of upper triangular matrices with ones on the diagonal.*

Note that as a group $\mathbb{G}_m$ is the set of units in K under multiplication and $\mathbb{G}_a$ is equal to K under addition, and that both groups are affine of dimension 1; in fact they are the only linear algebraic groups of dimension one, up to isomorphism.

Note that if we are given a linear algebraic group G, we get a Hopf algebra A. Indeed if A is the coordinate ring of G, then A is a K-algebra and there are maps

$$A \longrightarrow A \otimes A \qquad \text{and} \qquad A \longrightarrow A,$$

induced by the multiplication and inverse map for G.

It is not hard to see that the product of two algebraic groups is an algebraic group.

Definition 2.4. *The algebraic group $\mathbb{G}_m^k$ is called a* ***torus****.*

So a torus in algebraic geometry is just a product of copies of $\mathbb{G}_m$.

In fact one can define what it means to be a group scheme:

Definition 2.5. *Let $\pi\colon X \longrightarrow S$ be a morphism of schemes. We say that X is a* ***group scheme*** *over S, if there are three morphisms, $e\colon S \longrightarrow X$, $\mu\colon X \underset{S}{\times} X \longrightarrow X$ and $i\colon X \longrightarrow X$ over S which satisfy the obvious axioms.*

We can define a group scheme $\mathbb{G}_{m,\operatorname{Spec}\mathbb{Z}}$ over $\operatorname{Spec}\mathbb{Z}$, by taking

$$\operatorname{Spec}\mathbb{Z}[x, x^{-1}].$$

Given any scheme S, this gives us a group scheme $\mathbb{G}_{m,S}$ over S, by taking the fibre square. In the case when $S = \operatorname{Spec} k$, k an algebraically closed field, then $\mathbb{G}_{m,\operatorname{Spec} k}$ is $t(\mathbb{G}_m)$, the scheme associated to the quasi-projective variety $\mathbb{G}_m$. We will be somewhat sloppy and not be too careful to distinguish the two cases.

Similarly we can take

$$\mathbb{G}_{a,\operatorname{Spec}\mathbb{Z}} = \operatorname{Spec}\mathbb{Z}[x].$$

Definition 2.6. *Let G be an algebraic group and let X be a variety acted on by G, $\pi\colon G \times X \longrightarrow X$. We say that the action is* ***algebraic*** *if π is a morphism.*

For example the natural action of $\operatorname{PGL}_n(K)$ on $\mathbb{P}^n$ is algebraic, and all the natural actions of an algebraic group on itself are algebraic.

Definition 2.7. *We say that a quasi-projective variety X is a* ***toric variety*** *if X is irreducible and normal and there is a dense open subset U isomorphic to a torus, such that the natural action of U on itself extends to an action on the whole of X.*

For example, any torus is a toric variety. $\mathbb{A}^n_k$ is a toric variety. The natural torus is the complement of the coordinate hyperplanes and the natural action is as follows

$$((t_1, t_2, \dots, t_n), (a_1, a_2, \dots, a_n)) \longrightarrow (t_1a_1, t_2a_2, \dots, t_na_n).$$

More generally, $\mathbb{P}^n$ is a toric variety. The action is just the natural action induced from the action above. A product of toric varieties is toric.

One thing to keep track of are the closures of the orbits. For the torus there is one orbit. For affine space and projective space the closure of the orbits are the coordinate subspaces.

Definition 2.8. *Let M be a lattice and let $N = \operatorname{Hom}(M, \mathbb{Z})$ be the dual lattice.*

A ***strongly convex rational polyhedral cone*** $\sigma \subset N_{\mathbb{R}} = N \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ *is*

- *a* ***cone****, that is, if* $v \in \sigma$ *and* $\lambda \in \mathbb{R}$, $\lambda \geq 0$ *then* $\lambda v \in \sigma$*;*
- ***polyhedral****, that is,* σ *is the intersection of finitely many half spaces;*
- ***rational****, that is, the half spaces are defined by equations with rational coefficients;*
- ***strongly convex****, that is,* σ *contains no linear spaces other than the origin.*

A ***fan*** *in* N *is a set* F *of finitely many strongly convex rational polyhedra, such that*

- *every face of a cone in* F *is a cone in* F*, and*
- *the intersection of any two cones in* F *is a face of each cone.*

One can reformulate some of the parts of the definition of a strongly rational polyhedral cone. For example, σ is a polyhedral cone if and only if σ is the intersection of finitely many half spaces which are defined by homogeneous linear polynomials. σ is a strongly convex polyhedral cone if and only if σ is a cone over finitely many vectors which lie in a common half space (in other words a strongly convex polyhedral cone is the same as a cone over a polytope). And so on.

We will see that the set of toric varieties, up to isomorphism, are in bijection with fans, up to the action of $\mathrm{SL}(n, \mathbb{Z})$.

We first give the recipe of how to go from a fan to a toric variety. Suppose we start with σ. Form the dual cone

$$\check{\sigma} = \{\, u \in M_{\mathbb{R}} \mid \langle u, v \rangle \geq 0, v \in \sigma \,\}.$$

Now take the integral points,

$$S_\sigma = \check{\sigma} \cap M.$$

Then form the (semi)group algebra,

$$A_\sigma = K[S_\sigma].$$

Finally form the affine variety,

$$U_\sigma = \operatorname{Spec} A_\sigma.$$

Given a semigroup S, to form the semigroup algebra $K[S]$, start with a K-vector space with basis χ^u, as u ranges over the elements of S. Given u and $v \in S$ define the product

$$\chi^u \cdot \chi^v = \chi^{u+v},$$

and extend this linearly to the whole of $K[S]$.

Example 2.9. *For example, suppose we start with* $M = \mathbb{Z}^2$, σ *the cone spanned by* $(1,0)$ *and* $(0,1)$, *inside* $N_{\mathbb{R}} = \mathbb{R}^2$. *Then* $\check{\sigma}$ *is spanned by the same vectors in* $M_{\mathbb{R}}$. *Therefore* $S_\sigma = \mathbb{N}^2$, *the group algebra is* $\mathbb{C}[x,y]$ *and so we get* $\mathbb{A}^2$. *Similarly if we start with the cone spanned by* $e_1, e_2, \dots, e_n$ *inside* $N_{\mathbb{R}} = \mathbb{R}^n$ *then we get* $\mathbb{A}^n$.

Now suppose we start with $\sigma = \{0\}$ *in* $\mathbb{R}$. *Then* $\check{\sigma}$ *is the whole of* $M_{\mathbb{R}}$, S_σ *is the whole of* $M = \mathbb{Z}$ *and so* $\mathbb{C}[M] = \mathbb{C}[x, x^{-1}]$. *Taking* Spec *we get the torus* $\mathbb{G}_m$.

More generally we always get a torus of dimension n if we take the origin in $\mathbb{R}^n$. Note that if $\tau \subset \sigma$ is a face then $\check{\sigma} \subset \check{\tau}$ is also a face so that $U_\tau \subset U_\sigma$ is an open subset. In fact

Lemma 2.10. *Let* $\tau \subset \sigma \subset N_{\mathbb{R}}$ *be a face of the cone* σ.

Then we may find $u \in S_\sigma$ *such that*

(1) $\tau = \sigma \cap u^\perp$,
(2)
$$S_\tau = S_\sigma + \mathbb{Z}^+(-u),$$
(3) A_τ *is a localisation of* A_σ, *and*
(4) U_τ *is a principal open subset of* U_σ.

Proof. The fact that every face of a cone is cut out by a hyperplane is a standard fact in convex geometry and this is (1). For (2) note that the RHS is contained in the LHS by definition of a cone. If $w \in S_\tau$ then $w + p \cdot u$ is in $\check{\sigma}$ for any p sufficiently large. If we take p to be also an integer this shows that w belongs to the RHS.

Let χ^u be the monomial corresponding to u. (2) implies that A_τ is the localisation of A_σ along χ^u. This is (3) and (4) is immediate from (3). □

Since the cone $\{0\}$ is a face of every cone, the affine scheme associated to a cone always contains a torus, which is then dense. In particular the affine scheme associated to a cone is always irreducible.

Definition 2.11. *Let* $S \subset T$ *be a subsemigroup of the semigroup* T. *We say that* S *is* ***saturated*** *in* T *if whenever* $u \in T$ *and* $p \cdot u \in S$ *for some positive integer* p, *then* $u \in S$.

Given a subsemigroup $S \subset M$ saturation is always with respect to M.

Lemma 2.12. *Suppose that* $S \subset M$.

Then $K[S]$ *is integrally closed if and only if* S *is saturated.*

Proof. Suppose that $K[S]$ is integrally closed.

Pick $u \in M$ such that $v = p \cdot u \in S$ for some positive integer p. Let $b = \chi^u$ and $a = \chi^v \in K[S]$. Then

$$b^p = \chi^{pu} = \chi^v = a,$$

so that b is a root of the monic polynomial $x^p - a \in K[S][x]$. As we are assuming that $K[S]$ is integrally closed this implies that $b \in K[S]$ which implies that $u \in S$. Thus S is saturated.

Now suppose that S is saturated. As $K[S] \subset K[M]$ and the latter is integrally closed, the integral closure L of $K[S]$ sits in the middle, $K[S] \subset L \subset K[M]$. The torus acts naturally on $K[M]$ and this action fixes $K[S]$, so that it also fixes L. L is therefore a direct sum of eigenspaces, which are all one dimensional (a set of commuting diagonalisable matrices are simultaneously diagonalisable) that is L has a basis of the form χ^u, as u ranges over some subset of M. It suffices to prove that $u \in S$.

Since $b = \chi^u$ is integral over $K[S]$, we may find $k_1, k_2, \dots, k_p \in K[S]$ such that

$$b^p + k_1 b^{p-1} + \dots + k_p = 0.$$

We may assume that every term is in the same eigenspace as b^p. We may also assume that $k_p \neq 0$. As b^p and $k_p \neq 0$ belong to the same eigenspace, which is one dimensional, we get $b^p \in K[S]$. Thus $pu \in S$ and so $u \in S$ as S is saturated. Thus $b \in K[S]$ and $K[S]$ is integrally closed. □

Note that S_σ is automatically saturated, as $\check{\sigma}$ is a rational polyhedral cone. In particular U_σ is normal.

Example 2.13. *Suppose that we start with the semigroup S generated by 2 and 3 inside $M = \mathbb{Z}$. Then*

$$K[S] = K[t^2, t^3] = K[x, y]/\langle y^2 - x^3 \rangle.$$

Note that this does come with an action of $\mathbb{G}_m$;

$$(t, x, y) \longrightarrow (t^2 x, t^3 y).$$

However the curve $V(y^2 - x^3) \subset \mathbb{A}^2$ is not normal.

In fact some authorities drop the requirement that a toric variety is normal.

An action of the torus corresponds to a map of algebras

$$A_\sigma \longrightarrow A_\sigma \underset{K}{\otimes} A_0,$$

which is naturally the restriction of

$$A_0 \longrightarrow A_0 \underset{K}{\otimes} A_0.$$

It is straightforward to check that the restricted map does land in $A_\sigma \underset{K}{\otimes} A_0$.

Lemma 2.14 (Gordan's Lemma)**.** *Let $\sigma \subset \mathbb{M}_{\mathbb{R}}$ be a strongly convex rational cone.*

Then S_σ is a finitely generated semigroup.

Proof. Pick vectors $v_1, v_2, \dots, v_n \in S_\sigma$ which generate the cone $\check{\sigma}$. Consider the set

$$K = \{\, v \in M \mid v = \sum t_i v_i, t_i \in [0,1] \,\}.$$

Then K is compact. As M is discrete $K \cap M$ is finite. I claim that the elements of $K \cap M$ generate the semigroup S_σ. Pick $u \in S_\sigma$. Since $u \in \check{\sigma}$ and $v_1, v_2, \dots, v_n$ generate the cone, we may write

$$u = \sum \lambda_i v_i,$$

where $\lambda_i \in \mathbb{Q}$. Let $n_i = \llcorner \lambda_i \lrcorner$. Then

$$u - \sum n_i v_i = \sum (\lambda_i - \llcorner \lambda_i \lrcorner) v_i \in K \cap M.$$

As $v_1, v_2, \dots, v_n \in K \cap M$ the result follows. $\square$

Gordan's lemma (2.14) implies that U_σ is of finite type over K. So U_σ is an affine toric variety.

Example 2.15. *Suppose we start with the cone spanned by e_2 and $2e_1 - e_2$. The dual cone is the cone spanned by f_1 and $f_1 + 2f_2$. Generators for the semigroup are f_1, $f_1 + f_2$ and $f_1 + 2f_2$. The corresponding group algebra is $A_\sigma = K[x, xy, xy^2]$. Suppose we call $u = x$, $v = xy$ and $w = xy^2$. Then $v^2 = x^2y^2 = x(xy^2) = uw$. Therefore the corresponding affine toric variety is given as the zero locus of $v^2 - uw$ in $\mathbb{A}^3$.*

Given a fan F, we get a collection of affine toric varieties, one for every cone of F. It remains to check how to glue these together to get a toric variety. Suppose we are given two cones σ and τ belonging to F. The intersection is a cone ρ which is also a cone belonging to F. Since ρ is a face of both σ and τ there are natural inclusions

$$U_\rho \subset U_\sigma \qquad \text{and} \qquad U_\rho \subset U_\tau.$$

We glue U_σ to U_τ using the natural identification of the common open subset U_ρ. Compatibility of gluing follows automatically from the fact that the identification is natural and from the combinatorics of the fan.

It is clear that the resulting scheme is of finite type over the groundfield. Separatedness follows from:

Lemma 2.16. *Let σ and τ be two cones whose intersection is the cone ρ.*

If ρ is a face of each then the diagonal map
$$U_\rho \longrightarrow U_\sigma \times U_\tau,$$
is a closed embedding.

Proof. This is equivalent to the statement that the natural map
$$A_\sigma \otimes A_\tau \longrightarrow A_\rho,$$
is surjective. For this, one just needs to check that
$$S_\rho = S_\sigma + S_\tau.$$
One inclusion is easy; the RHS is contained in the LHS. For the other inclusion, one needs a standard fact from convex geometry, which is called the separation lemma: there is a vector $u \in S_\tau \cap S_{-\tau}$ such that simultaneously
$$\rho = \sigma \cap u^\perp \qquad \text{and} \qquad \rho = \tau \cap u^\perp.$$
By the first equality $S_\rho = S_\sigma + \mathbb{Z}(-u)$. As $u \in S_{-\tau}$ we have $-u \in S_\tau$ and so the LHS is contained in the RHS. □

So we have shown that given a fan F we can construct a normal variety $X = X(F)$. It is not hard to see that the natural action of the torus corresponding to the zero cone extends to an action on the whole of X. Therefore $X(F)$ is indeed a toric variety.

Let us look at some examples.

Example 2.17. *Suppose that we start with $M = \mathbb{Z}$ and we let F be the fan given by the three cones $\{0\}$, the cone spanned by e_1 and the cone spanned by $-e_1$ inside $N_\mathbb{R} = \mathbb{R}$. The two big cones correspond to $\mathbb{A}^1$. We identify the two $\mathbb{A}^1$'s along the common open subset isomorphic to K^*. Now the first $\mathbb{A}^1 = \operatorname{Spec} K[x]$ and the second is $\mathbb{A}^1 = \operatorname{Spec} K[x^{-1}]$. So the corresponding toric variety is $\mathbb{P}^1$ (if we have homogeneous coordinates $[X : Y]$ on $\mathbb{P}^1$ coordinates on U_0 are $x = X/Y$ and $y = Y/X = 1/x$).*

Now suppose that we start with three cones in $N_\mathbb{R} = \mathbb{R}^2$, σ_1, σ_2 and σ_3. We let σ_1 be the cone spanned by e_1 and e_2, σ_2 be the cone spanned by e_2 and $-e_1 - e_2$ and σ be the cone spanned by $-e_1 - e_2$ and e_1. Let F be the fan given as the faces of these three cones. Note that the three affine varieties corresponding to these three cones are all copies of $\mathbb{A}^2$. Indeed, any two of the vectors, e_1, e_2 and $-e_1 - e_2$ are a basis not *only of the underlying vector space but they also generate the standard lattice. We check how to glue two such copies of $\mathbb{A}^2$.*

The dual cone of σ_1 is the cone spanned by f_1 and f_2 in $M_{\mathbb{R}} = \mathbb{R}^2$. The dual cone of σ_2 is the cone spanned by $-f_1$ and $-f_1 + f_2$. So we have $U_1 = \operatorname{Spec} K[x, y]$ and $U_2 = \operatorname{Spec} K[x^{-1}, x^{-1}y]$. On the other hand, if we start with $\mathbb{P}^2$ with homogeneous coordinates $[X : Y : Z]$ and the two basic open subsets $U_0 = \operatorname{Spec} K[Y/X, Z/X]$ and $U_1 = \operatorname{Spec} K[X/Y, Z/Y]$, then we get the same picture, if we set $x = Y/X$, $y = Z/X$ (since then $X/Y = x^{-1}$ and $Z/Y = Z/X \cdot X/Y = yx^{-1}$). With a little more work one can check that we have $\mathbb{P}^2$.

More generally, suppose we start with $n+1$ vectors $v_1, v_2, \dots, v_{n+1}$ in $N_{\mathbb{R}} = \mathbb{R}^n$ which sum to zero such that the first n vectors $v_1, v_2, \dots, v_n$ span the standard lattice. Let F be the fan obtained by taking all the cones spanned by all subsets of at most n vectors. One can check that the resulting toric variety is $\mathbb{P}^n$.

Now suppose that we take the four vectors e_1, e_2, $-e_1$ and $-e_2$ in $N_{\mathbb{R}} = \mathbb{R}^2$ and let F be the fan consisting of all cones spanned by at most two vectors. Then we get four copies of $\mathbb{A}^2$. It is easy to check that the resulting toric variety is $\mathbb{P}^1 \times \mathbb{P}^1$. Indeed the top two fans glue together to get $\mathbb{P}^1 \times \mathbb{A}^1$ and so on.

We have already seen that cones correspond to open subsets. In fact cones also correspond (in some sort of dual sense) to closed subsets, the closure of the orbits. First observe that given a fan F, we can associate a closed point x_σ to any cone σ. To see this, observe that one can spot the closed points of U_σ using semigroups:

Lemma 2.18. *Let $S \subset M$ be a semigroup. Then there is a natural bijection,*

$$\operatorname{Hom}(K[S], K) \simeq \operatorname{Hom}(S, K).$$

Here the RHS is the set of semigroup homomorphisms, where $K = \{0\} \cup K^*$ is the multiplicative subsemigroup of K (and not the additive).

Proof. Suppose we are given a ring homomorphism

$$f \colon K[S] \longrightarrow K.$$

Define

$$g \colon S \longrightarrow K,$$

by sending u to $f(\chi^u)$. Conversely, given g, define $f(\chi^u) = g(u)$ and extend linearly. □

Consider the semigroup homomorphism:

$$S_\sigma \longrightarrow \{0, 1\},$$

where $\{0,1\} \subset \{0\} \cup K^*$ inherits the obvious semigroup structure. We send $u \in S_\sigma$ to 1 if $u \in \sigma^\perp$ and send it 0 otherwise. Note that as $\sigma^\perp$ is a face of $\check{\sigma}$ we do indeed get a homomorphism of semigroups. By (2.18) we get a surjective ring homomorphism

$$K[S_\sigma] \longrightarrow K.$$

The kernel is a maximal ideal of $K[S_\sigma]$, that is a closed point x_σ of U_σ, with residue field K.

To get the orbits, take the orbits of these points. It follows that the orbits are in correspondence with the cones in F. In fact the correspondence is inclusion reversing.

Example 2.19. *For the fan corresponding to $\mathbb{P}^1$, the point corresponding to $\{0\}$ is the identity, and the points corresponding to e_1 and $-e_1$ are 0 and ∞. For the fan corresponding to $\mathbb{P}^2$ the three maximal cones give the three coordinate points, the three one dimensional cones give the three coordinate lines (in fact the lines spanned by the points corresponding to the two maximal cones which contain them). As before the zero cone corresponds to the identity point. The orbit is the whole torus and the closure is the whole of $\mathbb{P}^2$.*

Suppose that we start with the cone σ spanned by e_1 and e_2 inside $N_{\mathbb{R}} = \mathbb{R}^2$. We have already seen that this gives the affine toric variety $\mathbb{A}^2$. Now suppose we insert the vector $e_1 + e_2$. We now get two cones σ_1 and σ_2, the first spanned by e_1 and $e_1 + e_2$ and the second spanned by $e_1 + e_2$ and e_2. Individually each is a copy of $\mathbb{A}^2$. The dual cones are spanned by f_2, $f_1 - f_2$ and f_1 and $f_2 - f_1$. So we get $\operatorname{Spec} K[y, x/y]$ and $\operatorname{Spec} K[x, x/y]$.

Suppose that we blow up $\mathbb{A}^2$ at the origin. The blow up sits inside $\mathbb{A}^2 \times \mathbb{P}^1$ with coordinates (x, y) and $[S : T]$ subject to the equations $xT = yS$. On the open subset $T \neq 0$ we have coordinates s and y and $x = sy$ so that $s = x/y$. On the open subset $S \neq 0$ we have coordinates x and t and $y = xt$ so that $t = y/x$. So the toric variety above is nothing more than the blow up of $\mathbb{A}^2$ at the origin. The central ray corresponds to the exceptional divisor E, a copy of $\mathbb{P}^1$.

A couple of definitions:

Definition 2.20. *Let G and H be algebraic groups which act on quasi-projective varieties X and Y. Suppose we are given an algebraic group homomorphism, $\rho\colon G \longrightarrow H$. We say that a morphism $f\colon X \longrightarrow Y$ is ρ-**equivariant** if f commutes with the action of G and H. If X and Y are toric varieties and G and H are the tori contained in X and Y then we say that f is a **toric morphism**.*

It is easy to see that the morphism defined above is toric. We can extend this picture to other toric surfaces. First a more intrinsic description of the blow up. Suppose we are given a ray, that is a one dimensional cone σ. Then we can describe σ by specifying the unique integral vector $v \in \sigma$ which is closest to the origin. Note that every other integral vector belonging to σ is a positive integral multiple of v, which we call the **primitive generator** v. Suppose we are given a toric surface and a two dimensional cone σ such that the primitive generators v and w of the two one dimensional faces of σ generate the lattice (so that up the action of $\mathrm{SL}(2, \mathbb{Z})$, σ is the cone spanned by e_1 and e_2). Then the blow up of the point corresponding to σ is a toric surface, which is obtained by inserting the sum $v + w$ of the two primitive generators and subdividing σ in the obvious way (somewhat like the barycentric subdivision in simplicial topology).

Example 2.21. *Suppose we start with $\mathbb{P}^2$ and the standard fan. If we insert the two vectors $-e_1$ and $-e_2$ this corresponds to blowing up two invariant points, say $[0 : 1 : 0]$ and $[0 : 0 : 1]$. Note that now $-e_1 - e_2$ is the sum of $-e_1$ and $-e_2$. So if we remove this vector this is like blowing down a copy of $\mathbb{P}^1$. The resulting fan is the fan for $\mathbb{P}^1 \times \mathbb{P}^1$.*

We can generalise this to higher dimensions. For example suppose we start with the standard cone for $\mathbb{A}^3$ spanned by e_1 and e_2 and e_3. If we insert the vector $e_1 + e_2 + e_3$ (thereby creating three maximal cones) this corresponds to blowing up the origin. (In fact there is a simple recipe for calculating the exceptional divisor; mod out by the central $e_1 + e_2 + e_3$; the quotient vector space is two dimensional and the three cones map to the three cones in the quotient two dimensional vector space which correspond to the fan for $\mathbb{P}^2$). Suppose we insert the vector $e_1 + e_2$. Then the exceptional locus is $\mathbb{P}^1 \times \mathbb{A}^1$. In fact this corresponds to blowing up one of the axes (the axis is a copy of $\mathbb{A}^1$ and over every point of the axis there is a copy of $\mathbb{P}^1$).